

# MORITA THEORY FOR COMODULES OVER CORINGS

GABRIELLA BÖHM AND JOOST VERCRUYSSÉ

**ABSTRACT.** By a theorem due to Kato and Ohtake, any (not necessarily strict) Morita context induces an equivalence between appropriate subcategories of the module categories of the two rings in the Morita context. These are in fact categories of firm modules for non-unital subrings. We apply this result to various Morita contexts associated to a comodule  $\Sigma$  of an  $A$ -coring  $\mathcal{C}$ . This allows to extend (weak and strong) structure theorems in the literature, in particular beyond the cases when any of the coring  $\mathcal{C}$  or the comodule  $\Sigma$  is finitely generated and projective as an  $A$ -module. That is, we obtain relations between the category of  $\mathcal{C}$ -comodules and the category of firm modules for a firm ring  $R$ , which is an ideal of the endomorphism algebra  $\text{End}^{\mathcal{C}}(\Sigma)$ . For a firmly projective comodule of a coseparable coring we prove a strong structure theorem assuming only surjectivity of the canonical map.

## INTRODUCTION

There is a long tradition of using Morita theory in the study of Hopf-Galois extensions and by generalization Galois corings and Galois comodules, see e.g. [16], [17], [1], [11], [5]. One of the applications of Galois theory within the context of corings and comodules is corresponding (generalized) descent theory. That is, a study of the adjoint Hom and tensor functors between the category of comodules over a coring and the category of modules over an appropriately chosen algebra. In particular, finding of sufficient and necessary conditions for these functors to be full and faithful. Results of this kind are referred to as (weak and strong) *structure theorems*. If a coring  $\mathcal{C}$  is finitely generated and projective as left  $A$ -module, then its category of right comodules becomes isomorphic to the category of modules over the dual ring  ${}^*\mathcal{C}$ . Hence Galois theory for such a coring describes in fact functors between two module categories, which explains the relation with Morita theory. Although in general Morita contexts for comodules can be constructed without any finiteness restriction on the coring  $\mathcal{C}$ , strictness of these Morita contexts implies such a finiteness condition for  $\mathcal{C}$ , and usually as well for the comodule  $\Sigma$ , appearing in the Morita context (see [5, Lemma 2.5]).

The standard result in Morita theory says that the connecting maps  $\nabla$  and  $\blacktriangledown$  of a Morita context  $(A, A', P, Q, \nabla, \blacktriangledown)$  are bijective (or equivalently surjective, if the algebras  $A$  and  $A'$  have a unit) if and only if the Morita context induces an equivalence of the module categories  $\mathcal{M}_A$  and  $\mathcal{M}_{A'}$ . It was a natural question to pose how distinct any Morita context is from an equivalence of categories. This question has in fact two categorically dual answers which were found by several authors. They say that any Morita context induces an equivalence between certain quotient categories of the original module categories (see [25]) as well as an equivalence between certain full

---

*Date:* May, 2008.

*1991 Mathematics Subject Classification.* 16D90, 16W30.

subcategories of the original module categories (see [23]). The occurring full subcategories consist of firm modules over the (possibly non-unital) rings  $P \triangleright Q$  and  $Q \blacktriangleright P$ . In this paper we will extensively use this latter result due to Kato and Ohtake [23].

Firm rings and firm modules did appear in coring theory as a tool to construct comatrix corings beyond the finitely generated and projective case. Recall that a (unital) bimodule (over unital rings) possesses a dual in the bicategory of bimodules – hence determines a comatrix coring – if and only if it is finitely generated and projective on the appropriate side. Without this finiteness property it may have a dual only in a larger bicategory. In [19] it was assumed that a bimodule has a dual in the bicategory of firm bimodules over firm rings and a Galois theory in this setting was developed.

The aim of this paper is to merge ideas of the Kato-Ohtake Theorem on equivalences between categories of firm modules induced by Morita contexts, with the application of Morita theory within the framework of comodules for corings. In relation with various questions, a number of Morita contexts has been associated to a comodule. Their strictness was shown to imply (weak and strong) structure theorems. Hereby we revisit some of these Morita contexts and derive corresponding structure theorems by means of the Kato-Ohtake Theorem. Since in this way strictness of the Morita context is no longer requested, we extend existing structure theorems in two ways. First, the coring  $\mathcal{C}$  will not have to be finitely generated and projective over its base ring  $A$ , and secondly, the comodule  $\Sigma$  will no longer have to be finitely generated and projective over  $A$ . As a consequence, our structure theorems relate the category of  $\mathcal{C}$ -comodules to a category of firm modules for a firm ring (instead of unital modules for a unital ring). That is, the framework developed in [19] is applied.

Another application of our theory is to coseparable corings. It is known that coseparable corings provide a class of examples of firm rings (see [8]). In particular, Galois theory for comodules over a coseparable coring can therefore be reduced to Morita theory between firm rings. Applying Morita theory over firm rings, in particular the Kato-Ohtake Theorem, to this situation, we are able to prove stronger results than for arbitrary corings. Most importantly, we show that, for a firmly projective comodule of a coseparable coring, surjectivity of the canonical map implies its bijectivity and this condition is equivalent to a Strong Structure Theorem (see Theorem 2.18). This theorem improves [29, Corollary 9.4], [31, 5.7, 5.8], [11, Proposition 5.6] and is ultimately related to [27, Theorem I]. The proof of Theorem 2.18 does not make use of any projectivity property of the coring as a module over its base, thus it differs conceptually from the proofs in the papers cited above.

The paper is organized as follows. In the first section we study, and recall facts about, general Morita theory. In Section 1.1 we collect some properties of firm rings and study their relation with idempotent rings and corings. Section 1.2 is devoted to a full proof of the Kato-Ohtake Theorem (which is included for the sake of completeness) and some related results. In Section 1.3 we develop a technique to reduce a general Morita context to a strict Morita context over firm rings. In Section 1.4 properties of Morita contexts between firm rings are discussed. The results of the first section are applied to particular Morita contexts associated to comodules in Section 2. The theory of [19] can be applied if, for a given comodule  $\Sigma$  of an  $A$ -coring  $\mathcal{C}$ , one can find a firm ring  $R$  together with a ring morphism  $R \rightarrow \text{End}^{\mathcal{C}}(\Sigma)$ , such that  $\Sigma$  is  $R$ -firmly projective as a right  $A$ -module. If  $\Sigma$  is a finitely generated and projective

right  $A$ -module, then  $R$  can be taken equal to  $\text{End}^{\mathcal{C}}(\Sigma)$ . In Section 2.1 we consider a canonical Morita context associated to  $\Sigma$  as a right  $A$ -module and making use of it, we describe a situation when one can find such a firm ring  $R$  in a general setting. In [5], generalizing a construction in [11], we associated a Morita context to any  $\mathcal{C}$ -comodule  $\Sigma$ , which connects the endomorphism ring  $\text{End}^{\mathcal{C}}(\Sigma)$  with the dual ring  ${}^*\mathcal{C}$  of  $\mathcal{C}$ . Its strictness was related to structure theorems. Using the results in Section 1, we can weaken the assumptions made in [5]. That is, instead of assuming surjectivity of the connecting maps, we make only assumptions on properties of its range. We apply a similar philosophy to reconsider in Section 2.2 a Morita context associated to a pure coring extension in [5]. Recall that two objects in a pre-additive category determine a Morita context of the hom-sets. In Section 2.3 we use this method to associate a canonical Morita context to two comodules, and show how the structure theorems of these comodules are related. In particular, starting with one comodule  $\Sigma$ , in a favourable situation (see Theorem 2.11), we associate a second comodule  $B$  to it, for which the strong structure theorem holds. Comparing the resulting Morita context, determined by the two comodules  $\Sigma$  and  $B$ , with the Morita context associated to  $\Sigma$  in Section 2.1, we derive structure theorems for  $\Sigma$ . In the final Section 2.4 we consider a Strong Structure Theorem for a firmly projective comodule  $\Sigma$  over a coseparable coring  $\mathcal{C}$ .

**Notations and conventions** For any object  $X$  in a category  $\mathcal{A}$ , we denote the identity morphism on  $X$  again by  $X$ .

Throughout the paper a *ring* means a module  $R$  over a fixed commutative ring  $k$ , together with a multiplication, i.e. a  $k$ -module map  $R \otimes_k R \rightarrow R$  satisfying the associativity constraint. When there is no risk of confusion, multiplication will be denoted by juxtaposition of elements of  $R$ . In general, we do not assume that the multiplication admits a unit. In the case when it does, i.e. there is an element  $1_R$  in  $R$  such that  $r1_R = r = 1_R r$ , for all  $r \in R$ , then we say that  $R$  is a *ring with unit* or a *unital ring*. A right module for a non-unital ring  $R$  (over a commutative ring  $k$ ) is a  $k$ -module  $M$  together with a  $k$ -module map  $M \otimes_k R \rightarrow M$ ,  $m \otimes r \mapsto mr$ , satisfying the associativity condition  $m(rr') = (mr)r'$ , for  $m \in M$  and  $r, r' \in R$ . The category of all right  $R$ -modules is denoted by  $\widetilde{\mathcal{M}}_R$ . By convention, for a unital algebra  $R$  we consider unital modules only. That is, right  $R$ -modules  $M$ , such that  $m1_R = m$ , for all  $m \in M$ . The category of unital right modules of a unital ring  $R$  is denoted by  $\mathcal{M}_R$ . Hom-sets in  $\widetilde{\mathcal{M}}_R$  and also in  $\mathcal{M}_R$  will be denoted by  $\text{Hom}_R(-, -)$ . The categories  ${}_R\widetilde{\mathcal{M}}$  and  ${}_R\mathcal{M}$  of left  $R$ -modules are defined symmetrically, and hom-sets are denoted as  ${}_R\text{Hom}(-, -)$ . The categories of  $R$ -bimodules will be denoted by  ${}_R\widetilde{\mathcal{M}}_R$  and  ${}_R\mathcal{M}_R$ , respectively, with hom-sets  ${}_R\text{Hom}_R(-, -)$ .

As in [19], the term *ideal* will be slightly abused in the following sense. Let  $\iota : R \rightarrow T$  be a morphism of (possibly non-unital) rings. If  $R$  is a left  $T$ -module such that  $\iota$  is left  $T$ -linear with respect to this action, then we will call  $R$  a left ideal for  $T$ , even if  $\iota$  is not necessarily injective. In particular, [19, Lemma 5.10] applies to this situation.

Let  $A$  be a ring with unit. An  $A$ -coring is a coalgebra (comonoid) in the monoidal category  ${}_A\mathcal{M}_A$ , i.e. a triple  $(\mathcal{C}, \Delta, \varepsilon)$ , where  $\mathcal{C}$  is an  $A$ -bimodule and the *coproduct*  $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}$ ,  $\Delta(c) =: c^{(1)} \otimes_A c^{(2)}$  (Sweedler notation, with implicit summation understood) and the *counit*  $\varepsilon : \mathcal{C} \rightarrow A$  are  $A$ -bimodule maps that satisfy  $(\Delta \otimes_A \mathcal{C}) \circ \Delta(c) = (\mathcal{C} \otimes_A \Delta) \circ \Delta(c) =: c^{(1)} \otimes_A c^{(2)} \otimes_A c^{(3)}$  and  $c^{(1)}\varepsilon(c^{(2)}) = c = \varepsilon(c^{(1)})c^{(2)}$ , for all  $c \in \mathcal{C}$ . A right  $\mathcal{C}$ -comodule consists of a right  $A$ -module  $M$  together with a right  $A$ -linear

map  $\rho^M : M \rightarrow M \otimes_A \mathcal{C}$ ,  $\rho^M(m) =: m^{[0]} \otimes_A m^{[1]}$  (with implicit summation), called a *coaction*, that satisfies  $(M \otimes_A \Delta) \circ \rho^M(m) = (\rho^M \otimes_A \mathcal{C}) \circ \rho^M(m) =: m^{[0]} \otimes_A m^{[1]} \otimes_A m^{[2]}$  and  $m^{[0]} \varepsilon(m^{[1]}) = m$ , for all  $m \in M$ . For two right  $\mathcal{C}$ -comodules  $M$  and  $N$ , a right  $A$ -module map  $M \rightarrow N$  is said to be *right  $\mathcal{C}$ -colinear* if  $\rho^N \circ f = (f \otimes \mathcal{C}) \circ \rho^M$ . The category of all right  $\mathcal{C}$ -comodules and right  $\mathcal{C}$ -colinear maps will be denoted as  $\mathcal{M}^{\mathcal{C}}$  and its hom-sets will be denoted by  $\text{Hom}^{\mathcal{C}}(-, -)$ . The category  ${}^{\mathcal{C}}\mathcal{M}$  of left  $\mathcal{C}$ -comodules, with hom-sets  ${}^{\mathcal{C}}\text{Hom}(-, -)$ , is defined symmetrically. For the coaction  $\rho^M$  on a left  $\mathcal{C}$ -comodule  $M$  the index notation  $\rho^M(m) = m^{[-1]} \otimes_A m^{[0]}$  is used, for  $m \in M$ . Let  $\mathcal{C}$  be an  $A$ -coring and  $R$  any (not necessarily unital) ring. If  $M \in \mathcal{M}^{\mathcal{C}}$  and  $M$  is a left  $R$ -module with multiplication map  $\mu : R \otimes M \rightarrow M$  such that  $\mu \in \mathcal{M}^{\mathcal{C}}$ , then we say that  $M$  is an  *$R$ - $\mathcal{C}$  bicomodule*, denoted by  $M \in {}_R\mathcal{M}^{\mathcal{C}}$ . For an extensive study of corings and comodules we refer to the monograph [9].

## 1. MORITA THEORY

**1.1. Firm modules.** In this first section of somewhat preliminary nature, we collect some facts about firm rings and their firm modules.

Let  $R$  be ring, not necessarily having a unit. The Dorroh-extension of  $R$  is a ring with unit:  $\hat{R} = R \oplus k$ . Moreover,  $\widetilde{\mathcal{M}}_R$  is isomorphic to the category  $\mathcal{M}_{\hat{R}}$  of unital  $\hat{R}$ -modules. The ring  $R$  is a two-sided ideal in  $\hat{R}$  and for all  $M \in \widetilde{\mathcal{M}}_R$  and  $N \in {}_R\widetilde{\mathcal{M}}$ ,

$$M \otimes_R N \cong M \otimes_{\hat{R}} N.$$

Let  $M$  be a right  $R$ -module. Then the right  $R$ -action on  $M$  induces a right  $R$ -linear morphism

$$\mu_{M,R} : M \otimes_R R \rightarrow M, \quad \mu_{M,R}(m \otimes_R r) = mr.$$

Denote  $MR := \{\sum_i m_i r_i \mid m_i \in M, r_i \in R\}$ . Then obviously,  $\mu_{M,R}$  factorizes as

$$\mu_{M,R} : M \otimes_R R \xrightarrow{\mathbf{m}_{M,R}} MR \xrightarrow{i} M,$$

where  $\mathbf{m}_{M,R}$  is surjective and  $i$  is the obvious inclusion map. Therefore,  $MR = M$  if and only if  $\mu_{M,R}$  is surjective. A ring  $R$  is said to be *idempotent* if and only if  $R^2 := RR = R$ .

For an arbitrary ring  $R$ , a right  $R$ -module  $M$  is called *firm* if  $\mu_{M,R}$  is an isomorphism. In this case, the inverse of  $\mu_{M,R}$  will be denoted by

$$\mathbf{d}_{M,R} : M \rightarrow M \otimes_R R, \quad \mathbf{d}_{M,R}(m) = m^r \otimes_R r,$$

with implicit summation understood. The category of all firm right  $R$ -modules with right  $R$ -linear maps between them is denoted by  $\mathcal{M}_R$ . (This notation is justified by the fact that a module  $M$  of a unital ring  $R$  is firm if and only if it is unital.) In the same way, we introduce the category  ${}_R\mathcal{M}$  of firm left  $R$ -modules and left  $R$ -linear maps and the category  ${}_R\mathcal{M}_S$  of firm bimodules where  $S$  is another ring. Taking  $M = R$ , we find  $\mu_R := \mu_{M,R} = \mu_{R,M}$ . Hence  $R \in \mathcal{M}_R$  if and only if  $R \in {}_R\mathcal{M}$ , i.e.  $\mu_R$  is an isomorphism with inverse denoted by  $\mathbf{d}_R$ . In this situation  $R$  is called a *firm ring*. This terminology is due to Quillen [26]. Examples of firm rings are rings with unit, rings with local units and coseparable corings (hence they can be constructed from split or separable extensions of (unital) rings [8]). Clearly, firm rings are idempotent, but the converse is not true. We do have, however, the following result, extending [24,

Proposition 2.5 (1)]. Note that, for any non-unital ring  $R$ , also  $R \otimes_R R$  is a non-unital ring, with multiplication

$$(1.1) \quad (r_1 \otimes_R r'_1)(r_2 \otimes_R r'_2) = r_1 r'_1 \otimes_R r_2 r'_2.$$

**Theorem 1.1.** *Let  $R$  be a ring (not necessarily with unit) and put  $S := R \otimes_R R$  which is a ring with multiplication (1.1). If  $R$  is idempotent, then the following statements hold.*

- (i) *If  $M \in \widetilde{\mathcal{M}}_R$  such that  $MR = M$ , then  $M \otimes_R R \in \mathcal{M}_R$  (cf. [24, Proposition 2.5 (1)]);*
- (ii) *For  $M \in \widetilde{\mathcal{M}}_R$  and  $N \in {}_R \widetilde{\mathcal{M}}$ , there is an isomorphism of  $k$ -modules  $M \otimes_R N \cong M \otimes_S N$ ;*
- (iii)  *$S$  is a firm ring;*
- (iv) *The categories  $\mathcal{M}_R$  and  $\mathcal{M}_S$  are canonically isomorphic;*
- (v) *If  $M \in \widetilde{\mathcal{M}}_R$  then  $MR \otimes_R R \in \mathcal{M}_S$ ;*
- (vi) *For any  $M \in \widetilde{\mathcal{M}}_R$ ,  $MR \otimes_R R \cong M \otimes_S S$ , as firm right  $S$ -modules.*

*Proof.* (i). First remark that associativity implies  $\mu_{M,R} \otimes_R R = M \otimes_R \mu_R$ . Consider the following exact row in  $\widetilde{\mathcal{M}}_R$ .

$$0 \longrightarrow \text{Ker } \mu_{M,R} \xrightarrow{i} M \otimes_R R \xrightarrow{\mu_{M,R}} M \longrightarrow 0.$$

Since the functor  $-\otimes_R R : \widetilde{\mathcal{M}}_R \rightarrow \widetilde{\mathcal{M}}_R$  is right exact, we find the following exact row in  $\widetilde{\mathcal{M}}_R$

$$(\text{Ker } \mu_{M,R}) \otimes_R R \xrightarrow{i \otimes_R R} M \otimes_R R \otimes_R R \xrightarrow{\mu_{M,R} \otimes_R R} M \otimes_R R \longrightarrow 0.$$

If we can show that  $(\text{Ker } \mu_{M,R}) \otimes_R R = 0$ , then  $\mu_{M,R} \otimes_R R$  is an isomorphism and therefore  $M \otimes_R R$  is a firm right  $R$ -module. Take  $\sum_j m_j \otimes_R r_j \otimes_R s \in (\text{Ker } \mu_{M,R}) \otimes_R R$ . Since  $R$  is idempotent, we can write  $s = \sum_i s_i s'_i \in R^2$ . Hence

$$\sum_j m_j \otimes_R r_j \otimes_R s = \sum_{i,j} m_j \otimes_R r_j \otimes_R s_i s'_i = \sum_{i,j} m_j \otimes_R r_j s_i \otimes_R s'_i = \sum_{i,j} m_j r_j \otimes_R s_i \otimes_R s'_i = 0.$$

Thus  $(\text{Ker } \mu_{M,R}) \otimes_R R = 0$  as needed.

(ii). If  $M \in \widetilde{\mathcal{M}}_R$ , then  $M \in \widetilde{\mathcal{M}}_S$  with action  $m \cdot (r \otimes_R r') = \mu_{M,R}(m \otimes_R r r') = m r r'$ , and similarly for  $N \in {}_R \widetilde{\mathcal{M}}$ . Take  $m \in M$ ,  $n \in N$  and  $r \otimes_R r' \in S$ , then

$$m \cdot (r \otimes_R r') \otimes_R n = m r r' \otimes_R n = m \otimes_R r r' n = m \otimes_R (r \otimes_R r') \cdot n.$$

Since  $R$  is idempotent, we can write any  $r \in R$  as  $r = \sum_i r_i r'_i$ . Therefore also

$$\begin{aligned} m r \otimes_S n &= \sum_i m r_i r'_i \otimes_S n = m \cdot \left( \sum_i r_i \otimes_R r'_i \right) \otimes_S n \\ &= m \otimes_S \left( \sum_i r_i \otimes_R r'_i \right) \cdot n = \sum_i m \otimes_S r_i r'_i n = m \otimes_S r n. \end{aligned}$$

These computations show that there exist unique morphisms  $j_1$  and  $j_2$  which render the following diagram commutative.

$$\begin{array}{ccccc}
 M \otimes R \otimes N & \xrightarrow{\quad\quad\quad} & M \otimes N & \xrightarrow{p_1} & M \otimes_R N \\
 & & \parallel & & \uparrow j_2 \downarrow j_1 \\
 M \otimes S \otimes N & \xrightarrow{\quad\quad\quad} & M \otimes N & \xrightarrow{p_2} & M \otimes_S N
 \end{array}$$

We find that  $j_2 \circ j_1 \circ p_1 = j_2 \circ p_2 = p_1$ . Since  $p_1$  is an epimorphism, we obtain that  $j_2 \circ j_1 = M \otimes_R N$ . In the same way,  $j_1 \circ j_2 = M \otimes_S N$ .

(iii). It follows from part (i) that  $R \otimes_R R \otimes_R R \cong R \otimes_R R$ . Therefore  $R \otimes_R R \otimes_R R \otimes_R R \cong R \otimes_R R$  as well. Moreover, by part (ii),  $(R \otimes_R R) \otimes_{R \otimes_R R} (R \otimes_R R) \cong (R \otimes_R R) \otimes_R (R \otimes_R R)$ . We conclude that  $S \otimes_S S \cong S$ , i.e.  $S$  is a firm ring.

(iv). Take  $M \in \mathcal{M}_R$ , then  $M \cong M \otimes_R R$  and hence  $M \otimes_R R \cong M \otimes_R R \otimes_R R = M \otimes_R S$ . By part (ii) also  $M \otimes_R S \cong M \otimes_S S$ . Combining these isomorphisms, we find that  $M \cong M \otimes_S S$ , i.e.  $M \in \mathcal{M}_S$ . Conversely, if  $M \in \mathcal{M}_S$ , then we can define a right  $R$ -action on  $M$  by  $m \cdot r = m^s \cdot (sr)$ , where  $\mathbf{d}_{M,S}(m) = m^s \otimes_S s \in M \otimes_S S$  is the unique element such that  $m^s \cdot s = m$ . Then  $M$  is firm as a right  $R$ -module by the following sequence of isomorphisms.

$$M \cong M \otimes_S S \cong M \otimes_S (S \otimes_R R) \cong (M \otimes_S S) \otimes_R R \cong M \otimes_R R.$$

(v). This follows immediately by (i) and (iv).

(vi). By part (i),  $MR \otimes_R R$  is a firm right  $R$ -module. By part (ii),  $MR \otimes_R S \cong MR \otimes_S S$ . Since  $R$  is idempotent by assumption,  $MR = MS$ .  $S$  is a firm ring by part (iii), hence the obvious map  $MS \otimes_S S \rightarrow M \otimes_S S$  has an inverse  $m \otimes_S ss' \mapsto ms \otimes_S s'$ . Thus the following sequence of right  $S$ -module isomorphisms holds.

$$MR \otimes_R R \cong MR \otimes_R R \otimes_R R \cong MR \otimes_R S \cong MR \otimes_S S \cong MS \otimes_S S \cong M \otimes_S S.$$

□

The following proposition provides a tool to construct idempotent rings, and therefore firm rings in combination with the previous theorem.

**Proposition 1.2.** *Let  $\mathcal{C}$  be an  $A$ -coring and  $T$  be an  $A$ -ring. If  $f$  is an idempotent element in the convolution algebra  ${}_A\mathrm{Hom}_A(\mathcal{C}, T)$ , then  $\mathrm{Im} f$  is an idempotent ring.*

*Proof.* Recall that multiplication in the convolution algebra  ${}_A\mathrm{Hom}_A(\mathcal{C}, T)$  is given by

$$(f * g)(c) = f(c^{(1)})g(c^{(2)}),$$

for all  $f, g \in {}_A\mathrm{Hom}_A(\mathcal{C}, T)$  and  $c \in \mathcal{C}$ . Hence  $f$  is idempotent in  ${}_A\mathrm{Hom}_A(\mathcal{C}, T)$  if and only if  $f(c) = f(c^{(1)})f(c^{(2)})$ , from which we immediately deduce that  $\mathrm{Im} f$  is an idempotent ring. □

**Example 1.3.** (i) Let  $\iota : R \rightarrow T$  be a ring morphism, where  $R$  is a firm ring.

We can regard  $R$  as an  $\hat{R}$ -coring (see [28, Lemma 2.1]), and  $\iota$  makes  $T$  into an  $R$ -ring. Multiplicativity of  $\iota$  corresponds exactly to the fact that  $\iota$  is an idempotent element of the convolution algebra  $_{\hat{R}}\mathrm{Hom}_{\hat{R}}(R, T)$ . Therefore  $\mathrm{Im} \iota$  is an idempotent ring, which can also easily be verified directly.

(ii) Let  $\mathcal{C}$  be an  $A$ -coring, then the counit  $\varepsilon \in {}_A\mathrm{Hom}_A(\mathcal{C}, A)$  is clearly idempotent. Hence  $R = \mathrm{Im} \varepsilon$  is an idempotent ring. In this situation there holds moreover a similar statement for the  $\mathcal{C}$ -comodules: for all  $M \in \mathcal{M}^{\mathcal{C}}$ , we have  $M \in \mathcal{M}_R$ .



*Remark 1.4.* Let  $R$  be a right ideal in a unital ring  $A$ . Regarding  $R$  as an  $R$ - $A$  bimodule, there is a functor

$$(1.2) \quad J_R := - \otimes_R R : \mathcal{M}_R \rightarrow \mathcal{M}_A.$$

Explicitly, for a firm right  $R$ -module  $M$  and  $m \in M$ , the action by  $a \in A$  on  $J_R(M) \cong M$  comes out as

$$(1.3) \quad m \cdot a = m^r(ra),$$

cf. [19, Lemma 5.11]. It is straightforward to check that restricting the  $A$ -action on  $J_R(M)$  to  $R$ , we recover the original  $R$ -module  $M$  (in particular,  $J_R(M)$  is firm as right  $R$ -module). That is to say, composing the functor  $J_R : \mathcal{M}_R \rightarrow \mathcal{M}_A$  with the forgetful functor  $\mathcal{M}_A \rightarrow \widetilde{\mathcal{M}}_R$ , we obtain the fully faithful inclusion functor  $\mathcal{M}_R \rightarrow \widetilde{\mathcal{M}}_R$ . Thus we conclude that both  $J_R$  and the forgetful functor  $\mathcal{M}_A \rightarrow \widetilde{\mathcal{M}}_R$  are fully faithful.

**Lemma 1.5.** *Let  $R$  be a right ideal in a (possibly non-unital) ring  $A$ . Then for any  $M \in \widetilde{\mathcal{M}}_A$  such that  $MR = M$ , there is a canonical isomorphism*

$$M \otimes_R P \cong M \otimes_A P, \quad \text{for all } P \in {}_A\widetilde{\mathcal{M}}.$$

*In particular, for any  $M \in \widetilde{\mathcal{M}}_A$ , the isomorphism  $M \cong M \otimes_A R$  holds if and only if  $M \cong M \otimes_R R$  holds.*

*Proof.* Take  $M \in \widetilde{\mathcal{M}}_A$  such that  $MR = M$ . Since the map  $\mu_{M,R} : M \otimes_R R \rightarrow M$  is surjective, we find for any  $m \in M$  a (not necessarily unique) element  $\sum_i m_i \otimes_R r_i \in M \otimes_R R$  such that  $\sum_i m_i r_i = m$ . Therefore, for all  $p \in P$  and  $a \in A$ ,

$$ma \otimes_R p = \sum_i m_i r_i a \otimes_R p = \sum_i m_i \otimes_R r_i a p = \sum_i m_i r_i \otimes_R a p = m \otimes_R a p.$$

Hence there exists a map  $M \otimes_A P \rightarrow M \otimes_R P$ ,  $m \otimes_A p \mapsto m \otimes_R p$ , which is easily seen to be the inverse of the epimorphism  $M \otimes_R P \rightarrow M \otimes_A P$ , induced by the inclusion  $R \rightarrow A$ .

Properties  $M \cong M \otimes_A R$  and  $M \cong M \otimes_R R$  of  $M \in \widetilde{\mathcal{M}}_A$  are equivalent since any of them implies that  $\mu_{M,R}$  is surjective.  $\square$

The following observation generalizes [28, Lemma 2.1 and Theorem 2.2].

**Theorem 1.6.** *An ideal  $R$  in a unital ring  $A$  is a firm ring if and only if  $R$  is an  $A$ -coring whose counit is the inclusion map  $R \rightarrow A$ . Moreover, if these equivalent conditions hold, then the category of firm right  $R$ -modules is isomorphic to the category of comodules over the  $A$ -coring  $R$ .*

*Proof.* Suppose first that  $R$  is a firm ring and define a coproduct  $\mathbf{d}_R : R \rightarrow R \otimes_R R \cong R \otimes_A R$ . It has a counit given by the inclusion  $R \rightarrow A$ . Conversely, if  $R$  is an  $A$ -coring with counit given by the inclusion  $R \rightarrow A$ , then its coproduct  $\Delta_R : R \rightarrow R \otimes_A R$ ,  $\Delta(r) = r_{(1)} \otimes_A r_{(2)}$  satisfies  $r = r_{(1)} r_{(2)}$ . This implies that  $\mu_R$  is surjective, i.e.  $R$  is an idempotent ring. Applying Lemma 1.5 we find that  $R \otimes_A R \cong R \otimes_R R$  and we can easily check that  $\Delta_R : R \rightarrow R \otimes_A R \cong R \otimes_R R$  is a two-sided inverse for  $\mu_R$ .

Take any  $M \in \mathcal{M}_A$ . Using Lemma 1.5, under the above conditions we see that a map  $M \rightarrow M \otimes_R R \cong M \otimes_A R$  is a counital coaction for the  $A$ -coring  $R$  if and only if it is inverse of the  $R$ -action  $M \otimes_R R \rightarrow M$ . Therefore an  $A$ -module map  $M \rightarrow M'$  is a morphism of firm  $R$ -modules if and only if it is a morphism of comodules.  $\square$

It follows by Theorem 1.6 that if  $R$  is a firm ring and an ideal in a unital ring  $A$ , then the functor (1.2) can be interpreted as the forgetful functor from the category of comodules for the  $A$ -coring  $R$  to  $\mathcal{M}_A$ . Hence we obtain

**Corollary 1.7.** *Let  $R$  be a firm ring that is an ideal in a unital ring  $A$ . Then the functor  $J_R : \mathcal{M}_R \rightarrow \mathcal{M}_A$  has a right adjoint given by  $- \otimes_R R \simeq - \otimes_A R : \mathcal{M}_A \rightarrow \mathcal{M}_R$ . Unit and counit are given, for all  $M \in \mathcal{M}_R$  and  $N \in \mathcal{M}_A$ , by*

$$\mathbf{d}_{M,R} : M \rightarrow J_R(M) \otimes_R R \quad \mu_{M,R} : J_R(N \otimes_R R) \rightarrow N.$$

Clearly  $\mathbf{d}_{M,R}$  is an isomorphism for all  $M \in \mathcal{M}_R$ , yielding another proof of fullness and faithfulness of  $J_R$  (cf. Remark 1.4).

**1.2. The Kato-Ohtake Theorem.** In this section we prove some results concerning Morita theory for general associative rings, with a focus on idempotent rings. This theory has been developed in a number of papers, see e.g. [23], [25]. Morita theory over firm rings has already been considered in [10] (where firm rings are named unital rings), [24] and [20] (where firm rings are named regular rings), however, some crucial points in the theory that will be of importance in this note are not treated in these papers.

Recall that a Morita context is a sextuple  $(A, A', P, Q, \nabla, \blacktriangledown)$ , consisting of two rings  $A$  and  $A'$  (with or without unit), two bimodules  $P \in {}_A \widetilde{\mathcal{M}}_{A'}$  and  $Q \in {}_{A'} \widetilde{\mathcal{M}}_A$  and two bilinear maps  $\nabla : P \otimes_{A'} Q \rightarrow A$  and  $\blacktriangledown : Q \otimes_A P \rightarrow A'$ , that are subjected to the following conditions

$$\begin{array}{ccc} P \otimes_{A'} Q \otimes_A P & \xrightarrow{P \otimes_{A'} \nabla} & P \otimes_{A'} A' \\ \nabla \otimes_A P \downarrow & & \downarrow \mu_{P,A'} \\ A \otimes_A P & \xrightarrow{\mu_{A,P}} & P \end{array} \quad \begin{array}{ccc} Q \otimes_A P \otimes_{A'} Q & \xrightarrow{Q \otimes_A \nabla} & Q \otimes_A A \\ \blacktriangledown \otimes_{A'} Q \downarrow & & \downarrow \mu_{Q,A} \\ A' \otimes_{A'} Q & \xrightarrow{\mu_{A',Q}} & Q \end{array}$$

The interest in Morita contexts arises from the fact that they can be used to study equivalences between categories. A first step is the following well-known theorem that relates a Morita context to a pair of functors between module categories, together with natural transformations relating these functors. A nice formulation of this theorem makes use of the notion of a *wide Morita context*, introduced in [14]. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two Abelian categories, then  $(F, G, \eta, \rho)$  is said to be a right wide Morita context between  $\mathcal{A}$  and  $\mathcal{B}$  if and only if  $F : \mathcal{A} \rightarrow \mathcal{B}$  and  $G : \mathcal{B} \rightarrow \mathcal{A}$  are right exact functors and  $\eta : GF \rightarrow 1_{\mathcal{A}}$  and  $\rho : FG \rightarrow 1_{\mathcal{B}}$  are natural transformations satisfying the conditions

$$F\eta = \rho F \quad \text{and} \quad G\rho = \eta G.$$

The following lemma extends [15, Proposition 5.2] about Morita contexts between unital rings.

**Lemma 1.8.** *Let  $A$  and  $A'$  be firm rings. Then there is a bijective correspondence between the following objects.*

- (i) *Morita contexts of the form  $(A, A', P, Q, \nabla, \blacktriangledown)$ , where  $P \in {}_A \mathcal{M}_{A'}$  and  $Q \in {}_{A'} \mathcal{M}_A$ ;*
- (ii) *Right wide Morita contexts  $(F, G, \omega, \beta)$  between  $\mathcal{M}_A$  and  $\mathcal{M}_{A'}$  such that  $F$  and  $G$  preserve direct sums;*
- (iii) *Right wide Morita contexts  $(F', G', \omega', \beta')$  between  ${}_A \mathcal{M}$  and  ${}_{A'} \mathcal{M}$  such that  $F$  and  $G$  preserve direct sums.*



*Proof.*  $(i) \Rightarrow (ii)$ . We can define functors  $F$  and  $G$  by  $F(M) = M \otimes_A P$  and  $G(N) = N \otimes_{A'} Q$  for all  $M \in \mathcal{M}_A$  and  $N \in \mathcal{M}_{A'}$ . The natural transformations  $\omega$  and  $\beta$  are given by

$$(1.4) \quad \omega_M : M \otimes_A P \otimes_{A'} Q \xrightarrow{M \otimes_A \nabla} M \otimes_A A \xrightarrow{\mu_{M,A}} M \quad \text{and}$$

$$(1.5) \quad \beta_N : N \otimes_{A'} Q \otimes_A P \xrightarrow{N \otimes_{A'} \blacktriangledown} N \otimes_{A'} A' \xrightarrow{\mu_{N,A'}} N.$$

$(ii) \Rightarrow (i)$ . By the Eilenberg-Watts Theorem (for the Eilenberg-Watts Theorem over firm rings we refer to [28]), we can write  $F \simeq - \otimes_A P$  with  $P \in {}_A \mathcal{M}_{A'}$  and  $G \simeq - \otimes_{A'} Q$  with  $Q \in {}_{A'} \mathcal{M}_A$ . Defining  $\nabla = \omega_A \circ (d_{A,P} \otimes_{A'} Q)$  and  $\blacktriangledown = \beta_{A'} \circ (d_{A',Q} \otimes_A P)$ , we easily find that  $(A, A', P, Q, \nabla, \blacktriangledown)$  is a Morita context.

The equivalence  $(iii) \Leftrightarrow (i)$  is proven symmetrically.  $\square$

The Kato-Ohtake Theorem says that, even without assuming that in a Morita context  $(A, A', P, Q, \nabla, \blacktriangledown)$  the rings  $A$  and  $A'$  are firm and their bimodules  $P$  and  $Q$  are firm, there are (equivalence) functors  $- \otimes_{\bar{A}} P : \mathcal{M}_{\bar{A}} \rightarrow \mathcal{M}_{\bar{A}'}$  and  $- \otimes_{\bar{A}'} Q : \mathcal{M}_{\bar{A}'} \rightarrow \mathcal{M}_{\bar{A}}$ , where  $\bar{A} := P \nabla Q$  and  $\bar{A}' := Q \blacktriangledown P$  are two-sided ideals in  $A$  and  $A'$ , respectively. Our next task is to recall this result. We first prove the following lemmata.

**Lemma 1.9.** *Let  $(F, G, \omega, \beta)$  be a right wide Morita context between the categories  $\mathcal{A}$  and  $\mathcal{B}$ . If  $\omega_A$  is an isomorphism for all  $A \in \mathcal{A}$  then  $(F, G)$  is an adjoint pair and  $F$  is a fully faithful functor.*

*Proof.* If  $\omega : GF \rightarrow \mathcal{A}$  is a natural isomorphism then  $\omega^{-1} : \mathcal{A} \rightarrow GF$  is the unit, while  $\beta : FG \rightarrow \mathcal{B}$  is the counit for the adjunction  $(F, G)$ . Since the unit  $\omega^{-1}$  of the adjunction is a natural isomorphism, the left adjoint  $F$  is fully faithful.  $\square$

**Lemma 1.10.** *Let  $(A, A', P, Q, \nabla, \blacktriangledown)$  be a Morita context between not necessarily unital rings, such that the connecting map  $\nabla$  is surjective. Then, for all  $M \in \widetilde{\mathcal{M}}_A$ , the morphism  $\omega_M$  in (1.4) is an isomorphism if and only if  $M \in \mathcal{M}_A$ .*

*Proof.* Suppose first that  $M$  is a firm right  $A$ -module. We have to show that  $\omega_M$  is an isomorphism. Since both  $\mu_{M,A}$  and  $\nabla$  are surjective,  $\omega_M$  is an epimorphism. Let us prove that  $\omega_M$  is also a monomorphism, i.e.  $\text{Ker } \omega_M = 0$ . To this end, consider the following commutative diagram in  $\widetilde{\mathcal{M}}_A$ .

$$\begin{array}{ccccccc} \text{Ker } \omega_M & \xrightarrow{\quad} & M \otimes_A P \otimes_{A'} Q & \xrightarrow{\quad \omega_M \quad} & M & \xrightarrow{\quad} & 0 \\ \uparrow \mu_{\text{Ker } \omega_M, A} & & \uparrow \mu_{M \otimes_A P \otimes_{A'} Q, A} & & \uparrow \mu_{M, A} & & \\ \text{Ker } \omega_M \otimes_A A & \xrightarrow{\quad} & M \otimes_A P \otimes_{A'} Q \otimes_A A & \xrightarrow{\quad \omega_M \otimes_A A \quad} & M \otimes_A A & \xrightarrow{\quad} & 0 \end{array}$$

The upper row is exact as  $\omega_M$  is an epimorphism and the exactness of lower row follows from the fact that the functor  $- \otimes_A A$  is right exact. Since  $M$  is firm as a right  $A$ -module,  $\mu_{M,A}$  is an isomorphism. Furthermore,  $\mu_{M \otimes_A P \otimes_{A'} Q, A}$  is surjective. Indeed, since  $\nabla$  is surjective, we can find for any element  $a \in A$ , a (not necessarily unique) element  $\sum p_a \otimes_{A'} q_a \in P \otimes_{A'} Q$  such that  $\sum p_a \nabla q_a = a$ . Hence, for all

$$m \otimes_A p \otimes_{A'} q \in M \otimes_A P \otimes_{A'} Q,$$

$$\begin{aligned}
\sum \mu_{M \otimes_A P \otimes_{A'} Q, A}(m^a \otimes_A p_a \otimes_{A'} q_a \otimes_A p \nabla q) &= \sum m^a \otimes_A p_a \otimes_{A'} q_a (p \nabla q) \\
&= \sum m^a \otimes_A p_a \otimes_{A'} (q_a \blacktriangledown p) q \\
&= \sum m^a \otimes_A p_a (q_a \blacktriangledown p) \otimes_{A'} q \\
&= \sum m^a \otimes_A (p_a \nabla q_a) p \otimes_{A'} q \\
&= m^a \otimes_A a p \otimes_{A'} q \\
&= m^a a \otimes_A p \otimes_{A'} q = m \otimes_A p \otimes_{A'} q.
\end{aligned}$$

A diagram chasing argument shows that surjectivity of  $\mu_{M \otimes_A P \otimes_{A'} Q, A}$  and injectivity of  $\mu_{M, A}$  imply surjectivity of  $\mu_{\text{Ker } \omega_M, A}$ . Hence  $\text{Ker } \omega_M = (\text{Ker } \omega_M)A$ . However,  $(\text{Ker } \omega_M)A$  contains only the zero element, as for any  $\sum_j m_j \otimes_A p_j \otimes_{A'} q_j \in \text{Ker } \omega_M$  and  $a = \sum p_a \nabla q_a \in A$  we find that

$$\begin{aligned}
\sum_j m_j \otimes_A p_j \otimes_{A'} q_j a &= \sum_{j, a} m_j \otimes_A p_j \otimes_{A'} q_j (p_a \nabla q_a) = \sum_{j, a} m_j (p_j \nabla q_j) \otimes_A p_a \otimes_{A'} q_a \\
&= \sum_{j, a} \omega_M(m_j \otimes_A p_j \otimes_{A'} q_j) \otimes_A p_a \otimes_{A'} q_a = 0.
\end{aligned}$$

Therefore  $\omega_M$  is an isomorphism.

Conversely, suppose now that  $\omega_M$  is an isomorphism. We need to show that  $M$  is a firm right  $A$ -module. For any  $m \in M$ ,

$$(1.6) \quad m = (\omega_M \circ \omega_M^{-1})(m) = \mu_{M, A} \circ (M \otimes_A \nabla) \circ \omega_M^{-1}(m).$$

Hence  $\mu_{M, A}$  is surjective, i.e.  $MA = M$ . Then also  $(M \otimes_A A)A = M \otimes_A A$ . Furthermore, by (1.6)  $\mu_{M, A}$  is a split epimorphism, proving that  $\text{Ker } \mu_{M, A}$  is a direct summand of the right  $A$ -module  $M \otimes_A A$ . Therefore,  $(\text{Ker } \mu_{M, A})A = \text{Ker } \mu_{M, A}$ . However, for all  $\sum_j m_j \otimes_A a_j \in \text{Ker } \mu_{M, A}$  and  $a' \in A$ , we find that  $\sum_j m_j \otimes_A a_j a' = \sum_j m_j a_j \otimes_A a' = 0$ . So we deduce that  $\text{Ker } \mu_{M, A} = (\text{Ker } \mu_{M, A})A = 0$ . Thus  $\mu_{M, A}$  is injective as well.  $\square$

Symmetrically to Lemma 1.10 one can consider a Morita context  $(A, A', P, Q, \nabla, \blacktriangledown)$  of non-unital rings, such that the connecting map  $\blacktriangledown$  is surjective. Then, for  $N \in \widetilde{\mathcal{M}}_{A'}$ , the morphism  $\beta_N$  in (1.5) is an isomorphism if and only if  $N \in \mathcal{M}_{A'}$ .

*Remark 1.11.* Take a Morita context  $(A, A', P, Q, \nabla, \blacktriangledown)$  of *unital* rings. Lemma 1.10 can be applied in particular to the restricted Morita context  $(\bar{A} := P \nabla Q, A', \bar{P}, \bar{Q}, \bar{\nabla}, \bar{\blacktriangledown})$ , where  $\bar{A}$  is a two-sided ideal in  $A$ ,  $\bar{P}$  is an  $\bar{A}$ - $A'$  bimodule and  $\bar{Q}$  is an  $A'$ - $\bar{A}$  bimodule via the restricted  $\bar{A}$ -actions,  $\bar{\nabla} : \bar{P} \otimes_{A'} \bar{Q} \rightarrow \bar{A}$  is given by corestriction of  $\nabla$  and  $\bar{\blacktriangledown} : \bar{Q} \otimes_{\bar{A}} \bar{P} \rightarrow A'$  is equal to the composite of the epimorphism  $\bar{Q} \otimes_{\bar{A}} \bar{P} \rightarrow Q \otimes_A P$  with  $\blacktriangledown$ . Note that, for a firm right  $\bar{A}$ -module  $M$ , we know by Lemma 1.5 that  $M \otimes_A X \cong M \otimes_{\bar{A}} X$  for all  $X \in {}_A \widetilde{\mathcal{M}}$ . Therefore the natural morphisms  $\omega_M$  and  $\bar{\omega}_M$  in (1.4), corresponding to the original and restricted Morita contexts, are related by the

following commutative diagram.

$$\begin{array}{ccccc}
 \omega_M : M \otimes_A P \otimes_{A'} Q & \xrightarrow{M \otimes_A \nabla} & M \otimes_A \bar{A} & \longrightarrow & M \\
 \cong \downarrow & & \cong \downarrow & & \parallel \\
 \bar{\omega}_M : M \otimes_{\bar{A}} P \otimes_{A'} Q & \xrightarrow{M \otimes_{\bar{A}} \bar{\nabla}} & M \otimes_{\bar{A}} \bar{A} & \xrightarrow{\cong} & M.
 \end{array}$$

Thus we conclude by Lemma 1.10 that  $\omega_M$  is an isomorphism for all  $M \in \mathcal{M}_{\bar{A}}$ . Conversely, if  $\omega_M$  is an isomorphism then  $\mu_{M, \bar{A}}$  is a (split) epimorphism. Hence the vertical arrows in the above diagram are isomorphisms by Lemma 1.5. Therefore also  $\bar{\omega}_M$  is an isomorphism, so  $M$  is a firm  $\bar{A}$ -module by Lemma 1.10.

Symmetrically, for a Morita context  $(A, A', P, Q, \nabla, \blacktriangledown)$  one can consider the other restricted Morita context  $(A, \bar{A}' := Q \blacktriangledown P, \bar{P}, \bar{Q}, \bar{\nabla}, \bar{\blacktriangledown})$ , where  $\bar{P}$  is an  $A$ - $\bar{A}'$  bimodule and  $\bar{Q}$  is an  $\bar{A}'$ - $A$  bimodule via the restricted  $\bar{A}'$ -actions,  $\bar{\blacktriangledown} : \bar{Q} \otimes_A \bar{P} \rightarrow \bar{A}'$  is given by corestriction of  $\blacktriangledown$  and  $\bar{\nabla} : \bar{P} \otimes_{\bar{A}'} \bar{Q} \rightarrow A$  is equal to the composite of the epimorphism  $\bar{P} \otimes_{\bar{A}'} \bar{Q} \rightarrow P \otimes_{A'} P$  with  $\nabla$ . Then the morphism  $\beta_N$  in (1.5) is an isomorphism if and only if  $N \in \mathcal{M}_{\bar{A}'}$ . Clearly, iteration of the two constructions (in arbitrary order) yields a Morita context

$$(1.7) \quad (\bar{A}, \bar{A}', \bar{P}, \bar{Q}, \bar{\nabla}, \bar{\blacktriangledown}),$$

with surjective (but not necessarily bijective) connecting maps.

**Theorem 1.12.** [23, Theorem 2.5] *Let  $(A, A', P, Q, \nabla, \blacktriangledown)$  be a Morita context of unital rings and consider the restricted Morita context (1.7). Then there is an equivalence of categories*

$$\mathcal{M}_{\bar{A}} \xrightleftharpoons[-\otimes_{\bar{A}'} \bar{Q}]{-\otimes_{\bar{A}} \bar{P}} \mathcal{M}_{\bar{A}'}.$$

*Proof.* Consider the following diagram of functors

$$\begin{array}{ccc}
 \mathcal{M}_A & \xrightleftharpoons[-\otimes_{A'} Q]{-\otimes_A P} & \mathcal{M}_{A'} \\
 J_{\bar{A}} \uparrow & & \uparrow J_{\bar{A}'} \\
 \mathcal{M}_{\bar{A}} & & \mathcal{M}_{\bar{A}'}
 \end{array}$$

where  $J_{\bar{A}}$  and  $J_{\bar{A}'}$  are defined as in (1.2). Recall from Remark 1.4 that  $J_{\bar{A}}(M) = M$  as (firm) right  $\bar{A}$ -modules, for any  $M \in \mathcal{M}_{\bar{A}}$ . Hence we can apply Lemma 1.10 to the Morita context (1.7) to conclude that  $\bar{\omega}_{J_{\bar{A}}(M)} = \mu_{M, \bar{A}} \circ (M \otimes_{\bar{A}} \bar{\nabla}) : M \otimes_{\bar{A}} P \otimes_{\bar{A}'} Q \rightarrow M$  is an isomorphism of right  $\bar{A}$ -modules. Symmetrically,  $\bar{\beta}_{J_{\bar{A}'}(M')} := \mu_{M', \bar{A}'} \circ (M' \otimes_{\bar{A}'} \bar{\blacktriangledown})$  is an isomorphism, for all  $M' \in \mathcal{M}_{\bar{A}'}$ . In order to show that  $J_{\bar{A}}(M) \otimes_A P = M \otimes_{\bar{A}} \bar{P}$  is a firm right  $\bar{A}'$ -module for all  $M \in \mathcal{M}_{\bar{A}}$ , we construct the inverse for the multiplication map  $\mu_{M \otimes_{\bar{A}} \bar{P}, \bar{A}'} : M \otimes_{\bar{A}} P \otimes_{\bar{A}'} \bar{A}' \rightarrow M \otimes_{\bar{A}} P$  as

$$d_{M \otimes_{\bar{A}} \bar{P}, \bar{A}'} : M \otimes_{\bar{A}} P \xrightarrow{\bar{\omega}_{J_{\bar{A}}(M)}^{-1} \otimes_{\bar{A}} P} M \otimes_{\bar{A}} P \otimes_{\bar{A}'} Q \otimes_{\bar{A}} P \xrightarrow{M \otimes_{\bar{A}} P \otimes_{\bar{A}'} \bar{\blacktriangledown}} M \otimes_{\bar{A}} P \otimes_{\bar{A}'} \bar{A}'.$$

Thus we conclude that the functors  $-\otimes_{\bar{A}} \bar{P} = (-\otimes_A P) \circ J_{\bar{A}} : \mathcal{M}_{\bar{A}} \rightarrow \mathcal{M}_{\bar{A}'}$  and  $-\otimes_{\bar{A}'} \bar{Q} = (-\otimes_{A'} Q) \circ J_{\bar{A}'} : \mathcal{M}_{\bar{A}'} \rightarrow \mathcal{M}_{\bar{A}}$  are well-defined. Moreover,  $(-\otimes_{\bar{A}} \bar{P}, -\otimes_{\bar{A}'} \bar{Q}, \bar{\omega}, \bar{\beta})$  constitute a right wide Morita context between  $\mathcal{M}_{\bar{A}}$  and  $\mathcal{M}_{\bar{A}'}$ . Since  $\bar{\omega}$  and

$\overline{\beta}$  are natural isomorphisms, it follows by Lemma 1.9 that the context induces an equivalence of categories.  $\square$

*Remark 1.13.* By symmetry, any Morita context  $(A, A', P, Q, \nabla, \blacktriangledown)$  with restricted form  $(\overline{A}, \overline{A'}, \overline{P}, \overline{Q}, \overline{\nabla}, \overline{\blacktriangledown})$  in (1.7) induces as well an equivalence of categories

$$\overline{A}\mathcal{M} \xrightleftharpoons[\overline{P} \otimes_{\overline{A}'}]{\overline{Q} \otimes_{\overline{A}}} \overline{A}'\mathcal{M}.$$

**1.3. Reduction of a Morita context.** Let  $(A, A', P, Q, \nabla, \blacktriangledown)$  be a Morita context. In this section we extend the construction of an associated Morita context (1.7) with *surjective* connecting maps to appropriate (non-unital) *subrings* of  $P \nabla Q$  and  $Q \blacktriangledown P$ .

**Lemma 1.14.** *Let  $(A, A', P, Q, \nabla, \blacktriangledown)$  be a Morita context and let  $B \subseteq Q \blacktriangledown P$  be an idempotent left ideal. That is, assume that  $BB = B$  and  $Q \blacktriangledown PB \subseteq B$ . In terms of  $B$ , introduce the ideal  $W := PB \nabla Q = P \nabla BQ$  in  $A$ . Consider  $P$  as a  $W$ - $B$  bimodule and  $Q$  as a  $B$ - $W$  bimodule via restriction. The (non-unital) rings  $B$  and  $W$  obey the following properties.*

- (i)  $Q \blacktriangledown PB = B$ ;
- (ii)  $WPB = PB$  and  $BQW = BQ$ ;
- (iii)  $W$  is idempotent, that is,  $WW = W$ ;
- (iv)  $B' := QW \blacktriangledown P$  is an idempotent ideal in  $Q \blacktriangledown P$ , satisfying  $B' = BQ \blacktriangledown P$  and  $W = PB' \nabla Q$ .

*Proof.* (i) Using the assumptions that  $B$  is an idempotent ring (in the first equality) and that it is a left ideal (in the final inclusion), we obtain a sequence of inclusions  $B = BB \subseteq (Q \blacktriangledown P)B \subseteq B$ .

(ii) By construction of  $W$  and part (i), associativity of the Morita context implies  $WPB = (P \nabla BQ)PB = PB(Q \blacktriangledown PB) = PBB = PB$ . Symmetrically,  $BQW = BQ(P \nabla BQ) = B(Q \blacktriangledown PB)Q = BBQ = BQ$ .

(iii) Using part (ii), one deduces  $WW = WPB \nabla Q = PB \nabla Q = W$ .

(iv) Interchanging in part (iii) the role of  $A$  with  $A'$ ,  $P$  with  $Q$  and  $\blacktriangledown$  with  $\nabla$ , and replacing  $B$  by  $W$  and  $W$  by  $B'$ , we conclude that  $B'$  is an idempotent ideal. Moreover,

$$B' = QW \blacktriangledown P = Q(PB \nabla Q) \blacktriangledown P = (Q \blacktriangledown P)B(Q \blacktriangledown P) = BQ \blacktriangledown P,$$

where the last equality follows by part (i). Since  $B$  is an idempotent left ideal in  $Q \blacktriangledown P$ , we have  $B = BB \subseteq BQ \blacktriangledown P = B'$ . Hence  $W = PB \nabla Q \subseteq PB' \nabla Q$ . Conversely,

$$PB' \nabla Q = PB(Q \blacktriangledown P) \nabla Q = (PB \nabla Q)(P \nabla Q) \subseteq PB \nabla Q = W,$$

since  $W = PB \nabla Q$  is a (right) ideal in  $P \nabla Q$ .  $\square$

**Definition 1.15.** Let  $\mathbb{M} = (A, A', P, Q, \nabla, \blacktriangledown)$  be a Morita context and let  $B \subseteq Q \blacktriangledown P$  be an idempotent left ideal. Introduce the ideal  $W := PB \nabla Q$  in  $A$ . Consider  $P$  as a  $W$ - $B$  bimodule and  $Q$  as a  $B$ - $W$  bimodule via restriction. The  $B$ -reduced form of  $\mathbb{M}$  is the Morita context

$$(1.8) \quad \overline{\mathbb{M}}_B := (W, B, P \otimes_B B, B \otimes_B Q, \overline{\nabla}, \overline{\blacktriangledown}),$$

with connecting maps

$$\begin{aligned} \overline{\nabla} : P \otimes_B B \otimes_B B \otimes_B Q &\rightarrow W, & p \otimes_B b \otimes_B b' \otimes_B q &\mapsto pb \nabla b'q, \\ \overline{\blacktriangledown} : B \otimes_B Q \otimes_W P \otimes_B B &\rightarrow B, & b \otimes_B q \otimes_W p \otimes_B b' &\mapsto bq \blacktriangledown pb'. \end{aligned}$$

One could consider many variations of the conditions on  $B$ , imposed in Definition 1.15. For example,  $B$  can be an ideal with respect to a ring morphism  $\iota : B \rightarrow Q \blacktriangledown P$ . Of course we can replace  $B$  by an idempotent right ideal, or as well consider the  $W$ -reduced form of  $\mathbb{M}$  where  $W$  is an idempotent left ideal of  $P \blacktriangledown Q$ . It follows from the following lemma that these approaches lead to equivalent descriptions.

**Lemma 1.16.** *Let  $(A, A', P, Q, \blacktriangledown, \blacktriangledown)$  be a Morita context. Then the following statements are equivalent.*

- (i) *There exists an idempotent left ideal  $B \subseteq Q \blacktriangledown P$ , that is  $BB = B$  and  $Q \blacktriangledown PB \subseteq B$ ;*
- (ii) *There exists an idempotent two-sided ideal  $B' \subset Q \blacktriangledown P$ , that is  $B'B' = B'$  and  $Q \blacktriangledown PB' \subseteq B'$  and  $B'Q \blacktriangledown P \subseteq B'$ ;*
- (iii) *There exists a firm ring  $\tilde{B}$  together with a ring morphism  $\iota : \tilde{B} \rightarrow Q \blacktriangledown P$  such that  $\tilde{B}$  becomes a left ideal in  $Q \blacktriangledown P$ , that is  $\tilde{B}$  is a left  $Q \blacktriangledown P$ -module and  $\iota$  is left  $Q \blacktriangledown P$ -linear;*
- (iv) *There exists a firm ring  $\tilde{B}$  together with a ring morphism  $\iota : \tilde{B} \rightarrow Q \blacktriangledown P$  such that  $\tilde{B}$  is a left  $A'$ -module and  $\iota$  is left  $A'$ -linear;*
- (v) *There exists a firm ring  $\tilde{B}'$  together with a ring morphism  $\iota : \tilde{B}' \rightarrow Q \blacktriangledown P$  such that  $\tilde{B}'$  becomes a two sided ideal in  $P \blacktriangledown Q$ , that is  $\tilde{B}'$  is a  $Q \blacktriangledown P$ -bimodule and  $\iota$  is  $Q \blacktriangledown P$ -bilinear;*
- (vi) *All statements (i)-(v), where we interchange the roles of  $A$  and  $A'$ ,  $P$  and  $Q$ ,  $\blacktriangledown$  and  $\blacktriangledown$ .*

*Proof.*  $(i) \Rightarrow (ii)$ . Put  $B' := BQ \blacktriangledown P$ , as in Lemma 1.14 (iv).

$(ii) \Rightarrow (i)$ . Trivial.

$(i) \Rightarrow (iii)$ . We know by Theorem 1.1 that  $\tilde{B} := B \otimes_B B$  is a firm ring. The multiplication on  $B$  composed by the inclusion map  $B \subseteq Q \blacktriangledown P$  defines a ring map  $\iota : \tilde{B} \rightarrow Q \blacktriangledown P$ , which is clearly left  $Q \blacktriangledown P$ -linear.

$(iii) \Rightarrow (i)$ . Take  $B = \text{Im } \iota$ . Then we know by Example 1.3 (ii) that  $B$  is an idempotent ring. Since  $\iota$  is left  $Q \blacktriangledown P$ -linear,  $B$  is a left  $Q \blacktriangledown P$ -module.

$(iii) \Rightarrow (iv)$ . Follows from the facts that  $Q \blacktriangledown P$  is a (left) ideal in  $A'$  and  $\tilde{B}$  is a firm ring. Indeed, define an  $A'$ -action on  $\tilde{B}$  as  $a' \cdot \tilde{b} := (a' \iota(\tilde{b}')) \cdot \tilde{b} \tilde{b}'$ .

$(iv) \Rightarrow (iii)$ . Trivial.

$(ii) \Leftrightarrow (v)$ . Repeating the proof of  $(i) \Leftrightarrow (iii)$ , put  $\tilde{B}' = B' \otimes_{B'} B'$ .

$(vi)$ . Suppose  $B$  exists as in (i), then we know by Lemma 1.14 (iii) that  $W = PB \blacktriangledown Q$  is an idempotent two-sided ideal in  $P \blacktriangledown Q$ . This is the symmetric statement of (ii). The converse follows by applying the same symmetry again.  $\square$

The next theorem provides us with a criterion to identify maximal ones among idempotent rings  $B$  in Lemma 1.14.

**Theorem 1.17.** *If  $A$  is a left Artinian ring and  $I$  is an ideal in  $A$ , then there exists a maximal idempotent left ideal  $B \subset I$ .*

*Proof.* Put  $I_1 = I$ . Consider  $I_2 \subset I_1$  as the image of the multiplication map  $\mu_1 : I \otimes_A I \rightarrow I$ . Inductively, we define for all  $n \in \mathbb{N}$ ,  $I_n$  as the image of the multiplication map  $\mu_{n-1} : I \otimes_A I_{n-1} \rightarrow I_{n-1}$ . Clearly every  $I_n$  is a left ideal in  $I$ , hence also a left ideal in  $A$ . Since  $A$  is left Artinian, there exists an  $N \in \mathbb{N}$  such that  $I_N = I_{N+1}$ .

Putting  $B = I_N$ , we obtain an idempotent left ideal in  $I$ . For any idempotent subring  $B_0$  of  $I$ ,  $B_0 = B_0^N \subseteq I_N = B$ . Hence  $B$  is maximal in  $I$ .  $\square$

*Remark 1.18.* In Definition 1.15 we associated to a Morita context  $\mathbb{M} = (A, A', P, Q, \nabla, \blacktriangledown)$  a reduced Morita context  $\widetilde{\mathbb{M}}_B$  between idempotent rings with surjective connecting maps. Using results in Theorem 1.1, one can work equivalently with a Morita context of firm rings and their firm bimodules. That is, with the same notations as in Lemma 1.16, denote  $\widetilde{B} = B \otimes_B B$  and  $\widetilde{W} = W \otimes_W W$ . Instead of the Morita context (1.8), one may consider

$$(1.9) \quad (\widetilde{W}, \widetilde{B}, \widetilde{W} \otimes_W P \otimes_B \widetilde{B}, \widetilde{B} \otimes_B Q \otimes_W \widetilde{W}, \tilde{\nabla}, \tilde{\blacktriangledown}),$$

with connecting maps

$$\begin{aligned} \tilde{\nabla} &: W \otimes_W W \otimes_W P \otimes_B B \otimes_B B \otimes_B B \otimes_B Q \otimes_W W \otimes_W W \rightarrow W \otimes_W W, \\ &w_1 \otimes_W w_2 \otimes_W p \otimes_B b_1 \otimes_B b_2 \otimes_B b'_1 \otimes_B b'_2 \otimes_B q' \otimes_W w'_1 \otimes_W w'_2 \mapsto \\ &w_1 w_2 (p b_1 b_2 \nabla b'_1 b'_2 q') \otimes_W w'_1 w'_2, \\ \tilde{\blacktriangledown} &: B \otimes_B B \otimes_B Q \otimes_W W \otimes_W W \otimes_W W \otimes_W W \otimes_W P \otimes_B B \otimes_B B \rightarrow B \otimes_B B, \\ &b_1 \otimes_B b_2 \otimes_B q \otimes_W w_1 \otimes_W w_2 \otimes_W w'_1 \otimes_W w'_2 \otimes_W p' \otimes_B b'_1 \otimes_B b'_2 \mapsto \\ &b_1 b_2 \otimes_B (q w_1 w_2 \blacktriangledown w'_1 w'_2 p') b'_1 b'_2. \end{aligned}$$

A few comments relating the Morita contexts (1.8) and (1.9) are in order.

- (i) The two reduced forms (1.8) and (1.9) of a Morita context exist under the same equivalent conditions of Lemma 1.16.
- (ii) We know from Theorem 1.1 that  $\widetilde{B}$  and  $\widetilde{W}$  are firm rings,  $\widetilde{W} \otimes_W P \otimes_B \widetilde{B}$  is a firm  $\widetilde{W}$ - $\widetilde{B}$  bimodule and  $\widetilde{B} \otimes_B Q \otimes_W \widetilde{W}$  is a firm  $\widetilde{B}$ - $\widetilde{W}$  bimodule. Therefore, since the connecting maps  $\tilde{\nabla}$  and  $\tilde{\blacktriangledown}$  are surjective by construction, it follows by Lemma 1.10 that  $\widetilde{W} \otimes_{\widetilde{W}} \tilde{\nabla} \cong \tilde{\nabla}$  and  $\widetilde{B} \otimes_{\widetilde{B}} \tilde{\blacktriangledown} \cong \tilde{\blacktriangledown}$  are bijective, that is to say, the Morita context (1.9) is *strict*.
- (iii) The connecting maps in both reduced Morita contexts (1.8) and (1.9) are surjective by construction. Therefore, we obtain by Theorem 1.1 and Theorem 1.12 the following commutative diagram of category equivalences.

$$(1.10) \quad \begin{array}{ccc} \mathcal{M}_W & \begin{array}{c} \xleftarrow{-\otimes_W P \otimes_B B} \\ \xrightarrow{-\otimes_B B \otimes_B Q \simeq -\otimes_B Q} \end{array} & \mathcal{M}_B \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{M}_{\widetilde{W}} & \begin{array}{c} \xleftarrow{-\otimes_{\widetilde{W}} \widetilde{W} \otimes_W P \otimes_B \widetilde{B} \simeq -\otimes_W P \otimes_B B} \\ \xrightarrow{-\otimes_{\widetilde{B}} \widetilde{B} \otimes_B Q \otimes_W \widetilde{W} \simeq -\otimes_B Q} \end{array} & \mathcal{M}_{\widetilde{B}} \end{array}$$

That is, both reduced forms (1.8) and (1.9) of a Morita context induce (up to isomorphism of categories) the same equivalence.

- (iv) Let us use the notations in Lemma 1.14 (iv), i.e. put  $B' = QW \blacktriangledown P$ . Consider the  $B$ -reduced and the  $B'$ -reduced forms of  $\mathbb{M}$ , with  $W = PB \nabla Q = PB' \nabla Q = W'$ .



Then we have equivalences

$$\begin{array}{ccc} \mathcal{M}_{W'} & \xrightleftharpoons[-\otimes_{B'} Q]{-\otimes_{W'} P \otimes_{B'} B'} & \mathcal{M}_{B'} \\ \parallel & & \\ \mathcal{M}_W & \xrightleftharpoons[-\otimes_B Q]{-\otimes_W P \otimes_B B} & \mathcal{M}_B . \end{array}$$

In particular,

$$(1.11) \quad - \otimes_{B'} Q \otimes_W P \otimes_B B \simeq - \otimes_B B : \mathcal{M}_{B'} \rightarrow \mathcal{M}_B$$

is an equivalence. Consider now the  $W$ -reduced form  $(W, B', W \otimes_W P, Q \otimes_W W, \underline{\nabla}, \underline{\nabla})$  of  $\mathbb{M}$ . It induces equivalence functors

$$\mathcal{M}_W \xrightleftharpoons[-\otimes_{B'} Q \simeq -\otimes_{B'} Q \otimes_W W]{-\otimes_W P \simeq -\otimes_W P \otimes_{B'} B'} \mathcal{M}_{B'} .$$

Thus we find that the  $B'$ -reduced and  $W$ -reduced Morita contexts give rise to the same equivalences of categories. Hence also the  $B$ -reduced and  $W$ -reduced forms give rise to the same equivalences (upto (1.11)).

- (v) Two points should be noticed here, which will be of importance later in this paper. First, we were able to reduce our original Morita context to a strict Morita context (1.9) that induces an equivalence between categories of firm modules over firm rings, and the functors are induced by firm bimodules. Second, it is possible to represent (at least) one of the functors by the original (possibly non-firm) bimodule from the original Morita context.

**1.4. Morita contexts between firm rings.** We finish this section by extending some classical results in Morita theory (of unital rings) to the situation of Morita contexts over firm rings, which applies in particular to the reduced Morita context of the form (1.9).

**Theorem 1.19.** *Consider a Morita context  $\mathbb{M} = (A, A', P, Q, \nabla, \blacktriangledown)$ , such that  $A$  is a firm ring and the connecting map  $\nabla$  is surjective. Then the following statements hold.*

- (i)  $P \otimes_{A'} Q$  is an  $A$ -ring with multiplication  $(p \otimes_{A'} q)(p' \otimes_{A'} q') := p(q \blacktriangledown p') \otimes_{A'} q'$  and unit  $u : A \rightarrow P \otimes_{A'} Q$  satisfying  $\nabla \circ u = A$ ;
- (ii) The functor  $- \otimes_A P : \mathcal{M}_A \rightarrow \mathcal{M}_{A'}$  is fully faithful;
- (iii)  $A \otimes_A P \cong A \otimes_A {}_{A'}\text{Hom}(Q, A')$  and  $Q \otimes_A A \cong \text{Hom}_{A'}(P, A') \otimes_A A$ , as  $A$ - $A'$  bimodules and  $A'$ - $A$  bimodules, respectively;
- (iv) There is a natural isomorphism  $- \otimes_{A'} Q \otimes_A A \simeq \text{Hom}_{A'}(P, -) \otimes_A A$  of functors  $\mathcal{M}_{A'} \rightarrow \mathcal{M}_A$  and a natural isomorphism  $A \otimes_A P \otimes_{A'} - \simeq A \otimes_A {}_{A'}\text{Hom}(Q, -)$  of functors  ${}_{A'}\mathcal{M} \rightarrow {}_A\mathcal{M}$ ;
- (v)  $A$  is a left ideal in  $\text{End}_{A'}(P)$  and a right ideal in  ${}_{A'}\text{End}(Q)^{\text{op}}$ ;
- (vi)  $Q \otimes_A A$  is a generator in  $\mathcal{M}_A$  and  $A \otimes_A P$  is a generator in  ${}_{A'}\mathcal{M}$ ;
- (vii) If in addition  $P$  is a firm left  $A$ -module or  $Q$  is a firm right  $A$ -module, then  $\nabla$  is bijective.

*Proof.* (i). Since  $A$  is a firm ring, it follows by Lemma 1.10 that  $A \otimes_A \nabla : A \otimes_A P \otimes_{A'} Q \rightarrow A \otimes_A A$  is bijective. Hence there is an  $A$ -bimodule map

$$u := (\mu_{A,P} \otimes_{A'} Q) \circ (A \otimes_A \nabla)^{-1} \circ \text{d}_A : A \rightarrow P \otimes_{A'} Q, \quad a \mapsto p_a \otimes_{A'} q_a,$$

with implicit summation understood. Since  $\nabla$  is left  $A$ -linear,

$$\nabla \circ u = \nabla \circ (\mu_{A,P} \otimes_{A'} Q) \circ (A \otimes_A \nabla)^{-1} \circ \mathbf{d}_A = \mu_A \circ (A \otimes_A \nabla) \circ (A \otimes_A \nabla)^{-1} \circ \mathbf{d}_A = A.$$

Moreover, for  $p \in P$ ,  $q \in Q$  and  $a \in A$ ,

$$\begin{aligned} p(q \blacktriangledown p_a) \otimes_{A'} q_a &= p \otimes_{A'} q(p_a \nabla q_a) = p \otimes_{A'} q a \quad \text{and} \\ p_a(q_a \blacktriangledown p) \otimes_{A'} q &= (p_a \nabla q_a)p \otimes_{A'} q = ap \otimes_{A'} q. \end{aligned}$$

Thus  $P \otimes_{A'} Q$  is an  $A$ -ring with the stated product and unit  $u$ .

(ii). Follows directly from Lemma 1.9 and Lemma 1.10.

(iii). Consider the  $A$ - $A'$  bimodule map

$$\alpha : A \otimes_A P \rightarrow A \otimes_A {}_{A'}\text{Hom}(Q, A'), \quad a \otimes_A p \mapsto a \otimes_A (- \blacktriangledown p).$$

In terms of the map  $u : a \mapsto p_a \otimes_{A'} q_a$  in part (i), its inverse is given by

$$\alpha^{-1} : A \otimes_A {}_{A'}\text{Hom}(Q, A') \rightarrow A \otimes_A P, \quad a_1 a_2 \otimes_A \varphi \mapsto a_1 \otimes_A p_{a_2} \varphi(q_{a_2}).$$

The other isomorphism  $Q \otimes_A A \cong \text{Hom}_{A'}(P, A') \otimes_A A$  follows by symmetrical reasoning.

(iv). A natural transformation  $- \otimes_{A'} Q \otimes_A A \rightarrow \text{Hom}_{A'}(P, -) \otimes_A A$  is given, for  $M \in \mathcal{M}_{A'}$ , by the right  $A$ -module map

$$\Phi_M : M \otimes_{A'} Q \otimes_A A \rightarrow \text{Hom}_{A'}(P, M) \otimes_A A, \quad m \otimes_{A'} q \otimes_A a \mapsto m(q \blacktriangledown -) \otimes_A a.$$

In terms of the map  $u : a \mapsto p_a \otimes_{A'} q_a$  in part (i), its inverse is given by

$$\Phi_M^{-1} : \text{Hom}_{A'}(P, M) \otimes_A A \rightarrow M \otimes_{A'} Q \otimes_A A, \quad \varphi \otimes_A a_1 a_2 \mapsto \varphi(p_{a_1}) \otimes_{A'} q_{a_1} \otimes_A a_2.$$

The other natural isomorphism  $A \otimes_A P \otimes_{A'} - \simeq A \otimes_A {}_{A'}\text{Hom}(Q, -)$  is proven symmetrically.

(v). As in the proof of part (i), Lemma 1.10 implies that  $\nabla \otimes_A A : P \otimes_{A'} Q \otimes_A A \rightarrow A \otimes_A A \cong A$  is an isomorphism. Moreover, by part (iv),  $P \otimes_{A'} Q \otimes_A A \cong \text{End}_{A'}(P) \otimes_A A$ . By [19, Lemma 5.10], the combined isomorphism  $\text{End}_{A'}(P) \otimes_A A \cong A$  means exactly that  $A$  is a left ideal in  $\text{End}_{A'}(P)$ . The other claim follows symmetrically.

(vi). To any  $p \in P$  we can associate a map  $f^p \in \text{Hom}_A(Q \otimes_A A, A)$ , defined by  $f^p(q \otimes_A a) := (p \nabla q)a$ . In terms of the map  $u : a \mapsto p_a \otimes_{A'} q_a$  in part (i),  $f^{p_a}(q_a \otimes_A \tilde{a}) = a\tilde{a}$ , for all  $a, \tilde{a} \in A$ . Hence it follows by the firm property of  $A$  that the evaluation map  $\text{Hom}_A(Q \otimes_A A, A) \otimes_{A'} Q \otimes_A A \rightarrow A$  is surjective. Consider the following commutative diagram with obvious maps, for all  $M \in \mathcal{M}_A$ .

$$\begin{array}{ccc} M \otimes_A \text{Hom}_A(Q \otimes_A A, A) \otimes_{A'} Q \otimes_A A & \longrightarrow & M \otimes_A A \\ \downarrow & & \downarrow \cong \\ \text{Hom}_A(Q \otimes_A A, M) \otimes_{A'} Q \otimes_A A & \longrightarrow & M \end{array}$$

Since the map in the upper row is an epimorphism, we find that the map in the lower row is an epimorphism as well, i.e.  $Q \otimes_A A$  is a generator for  $\mathcal{M}_A$ . It follows by a symmetrical reasoning that  $A \otimes_A P$  is a generator in  ${}_A\mathcal{M}$ .

(vii). Assume that  $P$  is a firm left  $A$ -module. Since  $\nabla$  is surjective, it follows from Lemma 1.10 that  $\omega_M$  is an isomorphism for all  $M \in \mathcal{M}_A$ , so for  $M = A$ . Since  $\omega_A$  and  $\nabla$  differ by isomorphisms  $(\mu_A$  and  $\mu_{A,P} \otimes_{A'} Q)$ ,  $\nabla$  is an isomorphism, too. The case when  $Q$  is a firm right  $A$ -module is treated symmetrically.  $\square$

Generalizing finitely generated projective modules over unital rings to firm modules over firm rings, the notion of firm projectivity was introduced in [28]. Since this notion plays a central role also in the present paper, in the next theorem we recall some facts about it (without proof).

**Theorem 1.20.** [28, Theorem 2.4],[29, Theorem 2.51] *Let  $R$  and  $S$  be firm rings and  $\Sigma$  a firm  $R$ - $S$  bimodule. The following statements are equivalent.*

- (i)  $\Sigma$  possesses a right dual (equal to  $\Sigma^* \otimes_R R$ ) in the bicategory of firm bimodules (formulated sometimes as  $\Sigma$  is a connecting bimodule in a comatrix coring context);
- (ii) There is a natural isomorphism  $\text{Hom}_S(\Sigma, -) \otimes_R R \simeq - \otimes_S \Sigma^* \otimes_R R$  of functors  $\mathcal{M}_S \rightarrow \mathcal{M}_R$ ;
- (iii) There is a non-unital ring map  $R \rightarrow \Sigma \otimes_S \Sigma^*$ , which induces the original left  $R$ -action on  $\Sigma$ .

Here  $\Sigma^* := \text{Hom}_S(\Sigma, S)$  and a (non-unital) multiplication in  $\Sigma \otimes_S \Sigma^*$  is induced by the evaluation map. A bimodule  $\Sigma$  obeying these equivalent properties is said to be an  $R$ -firmly projective right  $S$ -module.

**Corollary 1.21.** *Let  $\mathbb{M} = (A, A', P, Q, \nabla, \blacktriangledown)$  be a Morita context with unital rings  $A$  and  $A'$  and let  $R$  be a firm ring and left ideal in  $P \nabla Q$ . Then the following assertions hold.*

- (i) *There exists an  $R$ -bimodule map  $\Delta : R \rightarrow P \otimes_{A'} Q$ , such that  $(\nabla \circ \Delta)(r) = r$ , for all  $r \in R$ ;*
- (ii)  *$R \otimes_R P$  is an  $R$ -firmly projective right  $A'$ -module.*

*Proof.* The  $R$ -reduced Morita context  $(R, S := PR \nabla Q, R \otimes_R P, Q \otimes_R R, \bar{\nabla}, \bar{\blacktriangledown})$  satisfies all assumptions in Theorem 1.19. Hence by Theorem 1.19 (i), there is an  $R$ -bimodule map  $\bar{u} : R \rightarrow R \otimes_R P \otimes_S Q \otimes_R R$ , such that  $\bar{\nabla} \circ \bar{u} = R$ .

(i). In terms of  $\bar{u}$ , introduce the composite map

$$\Delta : R \xrightarrow{\bar{u}} R \otimes_R P \otimes_S Q \otimes_R R \xrightarrow{\mu_{R,P} \otimes_S \mu_{Q,R}} P \otimes_S Q \longrightarrow P \otimes_{A'} Q,$$

where the rightmost arrow denotes the canonical epimorphism. The map  $\Delta$  is  $R$ -bilinear and satisfies  $(\nabla \circ \Delta)(r) = (\bar{\nabla} \circ \bar{u})(r) = r$ , for all  $r \in R$ .

(ii).  $R \otimes_R P$  is a firm  $R$ - $A'$  bimodule hence the claim is proven by construction of a (non-unital) ring map  $j' : R \rightarrow (R \otimes_R P) \otimes_{A'} (R \otimes_R P)^*$ , which is compatible with the left  $R$ -action on  $R \otimes_R P$ . Consider the left  $A'$ -module map

$$f : Q \rightarrow (R \otimes_R P)^*, \quad q \mapsto (r \otimes_R p \mapsto q \blacktriangledown rp).$$

It can be used to construct an  $R$ -bimodule map

$$j' := (R \otimes_R P \otimes_{A'} f) \circ (R \otimes_R \Delta) \circ \text{d}_R : R \rightarrow (R \otimes_R P) \otimes_{A'} (R \otimes_R P)^*.$$

A straightforward computation yields  $j'(r_1)j'(r_2) = r_1 j'(r_2)$  for all  $r_1, r_2 \in R$  hence, by its left  $R$ -linearity, multiplicativity of  $j'$ . Its compatibility with the left  $R$ -action  $\mu_R \otimes_R P$  follows immediately by associativity of a Morita context and part (i).  $\square$

## 2. APPLICATIONS TO COMODULES OVER A CORING

In this section we apply the theory developed in Section 1 to various Morita contexts associated to comodules of corings. Let  $\mathcal{C}$  be a coring over a unital ring  $A$  and let  $\Sigma$  be a right  $\mathcal{C}$ -comodule. Denote  $T = \text{End}^{\mathcal{C}}(\Sigma)$ , then there exists a pair of adjoint functors

$$(2.1) \quad \mathcal{M}_T \begin{array}{c} \xrightarrow{-\otimes_T \Sigma} \\ \xleftarrow{\text{Hom}^{\mathcal{C}}(\Sigma, -)} \end{array} \mathcal{M}^{\mathcal{C}}$$

whose unit and counit are given by

$$(2.2) \quad \nu_N : N \rightarrow \text{Hom}^{\mathcal{C}}(\Sigma, N \otimes_T \Sigma), \quad \nu_N(n)(x) = n \otimes_T x;$$

$$(2.3) \quad \zeta_M : \text{Hom}^{\mathcal{C}}(\Sigma, M) \otimes_T \Sigma \rightarrow M, \quad \zeta_M(\varphi \otimes_T x) = \varphi(x);$$

for all  $N \in \mathcal{M}_T$  and  $M \in \mathcal{M}^{\mathcal{C}}$ . Galois theory for the comodule  $\Sigma$  includes the study of this pair of adjoint functors, in particular it concerns the question whether these functors or their (co)restrictions are fully faithful.

**2.1. Morita contexts associated to a comodule.** A first Morita context can be associated to a comodule  $\Sigma$  of an  $A$ -coring  $\mathcal{C}$  by considering it as right  $A$ -module. Then we can associate to  $\Sigma$  and  $\Sigma^* := \text{Hom}_A(\Sigma, A)$  a Morita context

$$(2.4) \quad (S = \text{End}_A(\Sigma), A, \Sigma, \Sigma^*, \nabla, \blacktriangledown),$$

as in [2, Section II.4]. The connecting maps are in this situation given by

$$(2.5) \quad \begin{array}{ll} \nabla : \Sigma \otimes_A \Sigma^* \rightarrow S, & x \nabla \xi = x\xi(-); \\ \blacktriangledown : \Sigma^* \otimes_S \Sigma \rightarrow A, & \xi \blacktriangledown x = \xi(x). \end{array}$$

If we put  $\bar{S} = \Sigma \nabla \Sigma^*$ , then we can restrict our Morita context to  $(\bar{S}, A, \Sigma, \Sigma^*, \bar{\nabla}, \blacktriangledown)$ , where we regard the restricted actions on  $\Sigma$  and  $\Sigma^*$ , the corestriction  $\bar{\nabla}$  of  $\nabla$  and the composite  $\blacktriangledown$  of the canonical epimorphism with  $\blacktriangledown$ . Since  $\bar{\nabla}$  is surjective by construction, we obtain by Lemma 1.9 and Lemma 1.10 an adjunction

$$(2.6) \quad \mathcal{M}_{\bar{S}} \begin{array}{c} \xrightarrow{-\otimes_{\bar{S}} \Sigma} \\ \xleftarrow{-\otimes_A \Sigma^*} \end{array} \mathcal{M}_A.$$

**Lemma 2.1.** *Let  $\iota : R \rightarrow S$  be a ring morphism where  $S$  is any (possibly non-unital, possibly non-firm) ring and  $R$  is a firm ring. Then the functor  $- \otimes_R S : \mathcal{M}_R \rightarrow \mathcal{M}_S$  has a right adjoint given by  $- \otimes_R R : \mathcal{M}_S \rightarrow \mathcal{M}_R$ .*

*Proof.* Consider  $S$  as an  $R$ -bimodule with actions given by  $r \cdot s \cdot r' = \iota(r)st(r')$  for all  $r, r' \in R$  and  $s \in S$ . Let us first check that the functor  $- \otimes_R S : \mathcal{M}_R \rightarrow \mathcal{M}_S$  is well-defined. Take any  $M \in \mathcal{M}_R$ , then  $M \otimes_R S$  is a firm right  $S$ -module, that is, the multiplication map

$$\mu : M \otimes_R S \otimes_S S \rightarrow M \otimes_R S, \quad \mu(m \otimes_R s \otimes_S t) = m \otimes_R st;$$

has a two-sided inverse

$$d : M \otimes_R S \rightarrow M \otimes_R S \otimes_S S, \quad d(m \otimes_R s) = m^r \otimes_R \iota(r) \otimes_S s.$$

Finally, let us give the unit  $\alpha$  and counit  $\beta$  for the adjunction, and leave other verifications to the reader.

$$\begin{aligned}\alpha_M : M &\rightarrow M \otimes_R S \otimes_R R, & \alpha_M(m) &= m^r \otimes_R \iota(r^t) \otimes_R t; \\ \beta_N : N \otimes_R R \otimes_R S &\rightarrow N, & \beta(n \otimes_R r \otimes_R s) &= n \cdot \iota(r)s,\end{aligned}$$

for all  $M \in \mathcal{M}_R$  and  $N \in \mathcal{M}_S$ .  $\square$

**Proposition 2.2.** *Let  $\mathcal{C}$  be a coring over a unital ring  $A$  and  $\Sigma$  be a right  $\mathcal{C}$ -comodule. Consider the Morita context (2.4) associated to  $\Sigma$  as right  $A$ -module and put  $\bar{S} = \Sigma \nabla \Sigma^*$  as before. If there exists a firm ring  $R$  together with a ring morphism  $\iota : R \rightarrow \bar{S}$ , then  $R \otimes_R \Sigma$  is  $R$ -firmly projective as a right  $A$ -module. If there exists moreover a ring morphism  $\iota' : R \rightarrow T = \text{End}^{\mathcal{C}}(\Sigma)$  then  $\Sigma$  (and therefore  $R \otimes_R \Sigma$  as well) is an  $R$ - $\mathcal{C}$  bicomodule.*

*Proof.* Combining the adjunction (2.6) with the adjunction in Lemma 2.1, we find that the functor  $-\otimes_R \bar{S} \otimes_{\bar{S}} \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}_A$  has a right adjoint given by  $-\otimes_A \Sigma^* \otimes_R R$ , cf.

$$\mathcal{M}_R \begin{array}{c} \xrightarrow{-\otimes_R \bar{S}} \\ \xleftarrow{-\otimes_R R} \end{array} \mathcal{M}_{\bar{S}} \begin{array}{c} \xrightarrow{-\otimes_{\bar{S}} \Sigma} \\ \xleftarrow{-\otimes_A \Sigma^*} \end{array} \mathcal{M}_A.$$

On the other hand, there is a natural isomorphism  $-\otimes_R \bar{S} \otimes_{\bar{S}} \Sigma \simeq -\otimes_R \Sigma$ , given for  $M \in \mathcal{M}_R$  by

$$m \otimes_R s \otimes_{\bar{S}} x \mapsto m \otimes_R sx, \quad \text{and} \quad m \otimes_R x \mapsto m^r \otimes_R \iota(r) \otimes_{\bar{S}} x.$$

Using the characterization of  $R$ -firmly projective modules in Theorem 1.20, we find that  $R \otimes_R \Sigma$  is  $R$ -firmly projective as a right  $A$ -module.

Clearly,  $\Sigma \in {}_R \mathcal{M}^{\mathcal{C}}$  if and only if the left action of  $R$  on  $\Sigma$  induces a ring morphism  $R \rightarrow \text{End}^{\mathcal{C}}(\Sigma)$ .  $\square$

Consider a coring  $\mathcal{C}$  over a unital ring  $A$  and a firm ring  $R$ . Any  $R$ - $A$ -bimodule  $\Sigma$  determines an adjunction

$$\mathcal{M}_R \begin{array}{c} \xrightarrow{-\otimes_R \Sigma} \\ \xleftarrow{\text{Hom}_A(\Sigma, -) \otimes_R R} \end{array} \mathcal{M}_A.$$

Since replacing the  $R$ - $A$  bimodule  $\Sigma$  by  $R \otimes_R \Sigma$  we obtain naturally isomorphic functors  $-\otimes_R \Sigma \simeq -\otimes_R R \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}_A$ , hence also the right adjoints are naturally isomorphic by

$$(2.7) \quad \chi_M : \text{Hom}_A(R \otimes_R \Sigma, M) \otimes_R R \rightarrow \text{Hom}_A(\Sigma, M) \otimes_R R.$$

If in addition  $\Sigma$  is an  $R$ - $\mathcal{C}$  bicomodule, then we have the following pair of adjoint functors

$$(2.8) \quad \mathcal{M}_R \begin{array}{c} \xrightarrow{F_{\Sigma} = -\otimes_R \Sigma \simeq -\otimes_R R \otimes_R \Sigma} \\ \xleftarrow{G_{\Sigma} = \text{Hom}^{\mathcal{C}}(R \otimes_R \Sigma, -) \otimes_R R \simeq \text{Hom}^{\mathcal{C}}(\Sigma, -) \otimes_R R} \end{array} \mathcal{M}^{\mathcal{C}}.$$

If  $R$  is equal to the unital ring  $\text{End}^{\mathcal{C}}(\Sigma)$  then (2.8) reduces to the adjunction (2.1). Since  $M \otimes_R R \otimes_R \Sigma \cong M \otimes_R \Sigma$  as right  $\mathcal{C}$ -comodules for all  $M \in \mathcal{M}_R$ , we find that the upper functors are naturally isomorphic, indeed. The natural isomorphism between

the lower functors follows from the uniqueness of the right adjoint. Unit and counit of the adjunction (2.8) are given explicitly by

$$(2.9) \quad \nu_N : N \rightarrow \text{Hom}^{\mathcal{C}}(\Sigma, N \otimes_R \Sigma) \otimes_R R, \quad \nu_N(n) = (x \mapsto n^r \otimes_R x) \otimes_R r,$$

$$(2.10) \quad \zeta_M : \text{Hom}^{\mathcal{C}}(\Sigma, M) \otimes_R \Sigma \rightarrow M, \quad \zeta_M(f \otimes_R x) = f(x),$$

for  $N \in \mathcal{M}_R$  and  $M \in \mathcal{M}^{\mathcal{C}}$ .

In any case when  $R \otimes_R \Sigma$  is an  $R$ - $\mathcal{C}$  bicomodule that is  $R$ -firmly projective as a right  $A$ -module, the theory developed in [19] can be applied to it. (We refer to [29, Section 4.2.5] for a more detailed treatment of structure theorems.) The above observations make it possible to translate occurring properties of  $R \otimes_R \Sigma$  to properties of  $\Sigma$ . Thus we obtain following

**Corollary 2.3.** *Let  $\mathcal{C}$  be a coring over a unital ring  $A$  and  $\Sigma$  be a right  $\mathcal{C}$ -comodule. Consider the Morita context (2.4) associated to  $\Sigma$  as right  $A$ -module and put  $\bar{S} = \Sigma \nabla \Sigma^*$  as before. Assume that there exists a firm ring  $R$  together with ring morphisms  $\iota : R \rightarrow \bar{S}$  and  $\iota' : R \rightarrow T = \text{End}^{\mathcal{C}}(\Sigma)$ . Then the following statements hold.*

- (i)  $\Sigma^\dagger := \Sigma^* \otimes_R R$  is a  $\mathcal{C}$ - $R$  bicomodule;
- (ii) There exists a comatrix coring  $\Sigma^\dagger \otimes_R \Sigma$  over  $A$ ;
- (iii) The map  $\text{can} : \Sigma^\dagger \otimes_R \Sigma \rightarrow \mathcal{C}$ ,  $\text{can}(\xi \otimes_R r \otimes_R x) = \xi(rx^{[0]})x^{[1]}$  is an  $A$ -coring morphism;
- (iv) The inner and outer triangles of the following diagram of adjoint functors commute (upto natural isomorphism)

$$\begin{array}{ccc}
 \mathcal{M}_R & \begin{array}{c} \xleftarrow{-\otimes_A \Sigma^\dagger} \\ \xrightarrow{-\otimes_R \Sigma} \end{array} & \mathcal{M}_A \\
 & \searrow \text{Hom}^{\mathcal{C}}(\Sigma, -) \otimes_R R & \downarrow \mathcal{F}^{\mathcal{C}} \\
 & & \mathcal{M}^{\mathcal{C}}
 \end{array}
 \quad \begin{array}{c} \uparrow -\otimes_A \mathcal{C} \\ \downarrow \end{array}$$

where  $\mathcal{F}^{\mathcal{C}}$  denotes the forgetful functor;

- (v) There is an adjunction  $(-\otimes_R \Sigma, -\otimes^{\mathcal{C}} \Sigma^\dagger)$ , where  $\otimes^{\mathcal{C}}$  denotes cotensor product over  $\mathcal{C}$ . Unit and counit of the adjunction are, for  $M \in \mathcal{M}^{\mathcal{C}}$  and  $N \in \mathcal{M}_R$ ,

$$\begin{aligned}
 N &\rightarrow (N \otimes_R \Sigma) \otimes^{\mathcal{C}} \Sigma^\dagger, & n &\mapsto (n^r)^{r'} \otimes_R e_{r'} \otimes_A (f_{r'} \otimes_R r), \\
 (M \otimes^{\mathcal{C}} \Sigma^\dagger) \otimes_R \Sigma &\rightarrow M, & m \otimes_A (f \otimes_R r) \otimes_R x &\mapsto mf(rx),
 \end{aligned}$$

where the map  $(\mu_{R, \Sigma} \otimes_A \mu_{\Sigma^*, R}) \circ (R \otimes_R \Sigma \otimes_A \chi_A) \circ (j \otimes_R R) \circ \mathbf{d}_R : R \rightarrow \Sigma \otimes_A \Sigma^*$ ,  $r \mapsto e_r \otimes_A f_r$  is obtained from the (non-unital) ring morphism  $j : R \rightarrow (R \otimes_R \Sigma) \otimes_A (R \otimes_R \Sigma)^*$ , coming from firm projectivity of  $R \otimes_R \Sigma$ , and the isomorphism  $\chi_A : (R \otimes_R \Sigma)^\dagger \rightarrow \Sigma^\dagger$ ,  $\varphi \otimes_R rr' \mapsto \varphi(r \otimes_R -) \otimes_R r'$  in (2.7).

By uniqueness of a right adjoint, there is a natural isomorphism  $-\otimes^{\mathcal{C}} \Sigma^\dagger \simeq \text{Hom}^{\mathcal{C}}(\Sigma, -) \otimes_R R$ ;

- (vi) If the functor  $\text{Hom}^{\mathcal{C}}(\Sigma, -) \otimes_R R : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_R$  is fully faithful then  $\text{can}$  is an isomorphism of  $A$ -corings;
- (vii) The functor  $\text{Hom}^{\mathcal{C}}(\Sigma, -) \otimes_R R : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_R$  is fully faithful and  $\mathcal{C}$  is flat as a left  $A$ -module if and only if  $\text{can}$  is an isomorphism and  $R \otimes_R \Sigma$  is flat as a left  $R$ -module (meaning that the functor  $-\otimes_R R \otimes_R \Sigma$ , from the category  $\mathcal{M}_{\bar{R}}$  of



- modules of the Dorroh-extension  $\hat{R}$  to the category  $Ab$  of Abelian groups, is (left) exact);
- (viii) If  $\Sigma$  is totally faithful as a left  $R$ -module (meaning that, for any  $N \in \mathcal{M}_R$ ,  $N \otimes_R \Sigma = 0$  implies  $N = 0$ ), then the functor  $- \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}^C$  is fully faithful. The converse holds if the map  $\Sigma \otimes_A \Sigma^\dagger \rightarrow \text{End}_A(\Sigma)$  is a pure left  $R$ -module monomorphism;
- (ix) If  $\mathcal{C}$  is flat as a left  $A$ -module, then  $- \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}^C$  is an equivalence if and only if  $\Sigma$  is faithfully flat as left  $R$ -module and can be an isomorphism of  $A$ -corings. Moreover, in this situation  $R$  is a left ideal in  $\text{End}^C(\Sigma)$ .

**Example 2.4.** Let  $\mathcal{C}$  be a coring over a unital ring  $A$  and let  $\Sigma$  be a right  $\mathcal{C}$ -comodule. In this example we provide an explicit construction of the firm ring  $R$  in Proposition 2.2 in appropriate situations. Let  $\bar{S} := \Sigma \nabla \Sigma^*$  be defined in terms of the connecting map (2.5), and  $T := \text{End}^C(\Sigma)$ . Put  $B := \bar{S} \cap T$ . Since  $\bar{S}$  is an ideal in  $S = \text{End}_A(\Sigma)$  by construction and  $T \subset S$ , we conclude that  $B$  is an ideal in  $T$ . If  $T$  is a (left) Artinian ring, then we can apply Theorem 1.17 to obtain an idempotent (left) ideal  $B' \subset B$ , which is still a (left) ideal in  $T$ , and a subring of  $\bar{S}$ . If we put now  $R := B' \otimes_{B'} B'$ , then  $R$  is a firm ring and the ring morphisms  $\iota : R = B' \otimes_{B'} B' \rightarrow \bar{S}$  and  $\iota' : R = B' \otimes_{B'} B' \rightarrow T$  are given by the multiplication on  $B'$ . Remark that  $R$  is still a left  $T$ -module.

In a recent paper [5] we associated also another Morita context to a comodule. Consider a coring  $\mathcal{C}$  over a unital ring  $A$  and a right  $\mathcal{C}$ -comodule  $\Sigma$ . There exists a Morita context

$$(2.11) \quad \mathbb{M}(\Sigma) = (T, {}^*\mathcal{C}, \Sigma, Q, \nabla, \blacktriangledown),$$

connecting the unital rings  $T := \text{End}^C(\Sigma)$  and  ${}^*\mathcal{C} = {}_A\text{Hom}(\mathcal{C}, A)$ . The bimodule  $Q$  is defined by

$$(2.12) \quad \begin{aligned} Q &= \{ q \in \text{Hom}_A(\Sigma, {}^*\mathcal{C}) \mid \forall x \in \Sigma, c \in \mathcal{C} \quad q(x^{[0]})(c)x^{[1]} = c^{(1)}q(x)(c^{(2)}) \} \\ &\cong \{ q \in {}_A\text{Hom}(\mathcal{C}, \Sigma^*) \mid \forall x \in \Sigma, c \in \mathcal{C} \quad c^{(1)}q(c^{(2)})(x) = q(c)(x^{[0]})x^{[1]} \}. \end{aligned}$$

The two forms of  $Q$  are related by interchanging the order of the arguments and their parallel use should cause no confusion. The connecting maps are

$$(2.13) \quad \blacktriangledown : Q \otimes_T \Sigma \rightarrow {}^*\mathcal{C}, \quad q \otimes_T x \mapsto q(x),$$

$$(2.14) \quad \nabla : \Sigma \otimes_{{}^*\mathcal{C}} Q \rightarrow T, \quad x \otimes_{{}^*\mathcal{C}} q \mapsto xq(-).$$

For more details we refer to [5, Section 2].

**Theorem 2.5.** For a unital ring  $A$ , let  $\mathcal{C}$  be an  $A$ -coring and  $\Sigma$  a right  $\mathcal{C}$ -comodule. Consider the Morita context (2.11) associated to  $\Sigma$ . If there exists a firm ring  $R$  together with a ring morphism  $\iota : R \rightarrow \Sigma \nabla Q$ , such that  $R$  is a left  $T$ -module and  $\iota$  is left  $T$ -linear, then the following statements hold.

- (i)  $\Sigma' := R \otimes_R \Sigma$  is an  $R$ -firmly projective right  $A$ -module;
- (ii) The functor  $- \otimes_R \Sigma \simeq - \otimes_R \Sigma' : \mathcal{M}_R \rightarrow \mathcal{M}^C$  is fully faithful;
- (iii) If moreover the functor  $\text{Hom}^C(\Sigma, -) : \mathcal{M}^C \rightarrow \mathcal{M}_T$  is fully faithful (as e.g. in the setting of [5, Theorem 4.1] or in forthcoming Theorem 2.7), then  $- \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}^C$  is an equivalence.

*Proof.* (i). This assertion follows immediately by Corollary 1.21 (ii).

(ii). Applying Theorem 1.19 (ii) to the  $R$ -reduced Morita context

$$(R, S := \Sigma R \vee Q \subseteq {}^*\mathcal{C}, R \otimes_R \Sigma, Q \otimes_R R, \overline{\vee}, \overline{\vee}),$$

one concludes that the functor  $F := - \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}_S$  is fully faithful. Moreover,  $F$  factorizes as

$$\mathcal{M}_R \xrightarrow{- \otimes_R \Sigma} \mathcal{M}^{\mathcal{C}} \longrightarrow \mathcal{M}_{\mathcal{C}^*} \longrightarrow \mathcal{M}_S.$$

The second and third functors act on the morphisms as the identity map, hence their composite is faithful, hence fully faithful. This proves that the leftmost arrow describes a full, hence fully faithful functor.

(iii). By part (ii) we know that the functor  $- \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}^{\mathcal{C}}$  is fully faithful. By assumption, also  $\text{Hom}^{\mathcal{C}}(\Sigma, -) : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_T$  is fully faithful. It was proven in [19] that this last statement is equivalent to the fact that the functor  $\text{Hom}^{\mathcal{C}}(\Sigma, -) \otimes_R R : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_R$  is fully faithful, since  $R$  is a left ideal in  $T$ .  $\square$

**2.2. A Morita context associated to a pure coring extension.** In this section we consider two corings  $\mathcal{D}$  and  $\mathcal{C}$  over unital rings  $L$  and  $A$ , respectively, such that  $\mathcal{C}$  is a  $\mathcal{C}$ - $\mathcal{D}$  bicomodule via the left regular  $\mathcal{C}$ -coaction (i.e.  $\mathcal{D}$  is a *right extension* of  $\mathcal{C}$  in the sense of [6]). Assume that  $\mathcal{D}$  is a *pure* coring extension of  $\mathcal{C}$ , in the sense that, for any right  $\mathcal{C}$ -comodule  $(M, \varrho)$ , the equalizer

$$M \xrightarrow{\varrho} M \otimes_A \mathcal{C} \xrightarrow[M \otimes_A \Delta_{\mathcal{C}}]{\varrho \otimes_A \mathcal{C}} M \otimes_A \mathcal{C} \otimes_A \mathcal{C}$$

in  $\mathcal{M}_L$  is  $\mathcal{D} \otimes_L \mathcal{D}$ -pure, i.e. it is preserved by the functor  $(-) \otimes_L \mathcal{D} \otimes_L \mathcal{D} : \mathcal{M}_L \rightarrow \mathcal{M}_L$ . Then, in addition to (2.4) and (2.11), we can associate to  $\Sigma \in {}_L\mathcal{M}^{\mathcal{C}}$  a further Morita context

$$(2.15) \quad \widetilde{\mathbb{M}}(\Sigma) = ({}_L\text{Hom}_L(\mathcal{D}, T), {}^{\mathcal{C}}\text{End}^{\mathcal{D}}(\mathcal{C})^{op}, {}_L\text{Hom}^{\mathcal{D}}(\mathcal{D}, \Sigma), \widetilde{Q}, \diamond, \blacklozenge).$$

Here  $T = \text{End}^{\mathcal{C}}(\Sigma)$ ,  ${}_L\text{Hom}_L(\mathcal{D}, T)$  is a convolution algebra and  ${}^{\mathcal{C}}\text{End}^{\mathcal{D}}(\mathcal{C})^{op}$  is the (opposite) endomorphism algebra of  $\mathcal{C}$  as  $\mathcal{C}$ - $\mathcal{D}$  bicomodule. The bimodule  $\widetilde{Q}$  is a subset of  $Q$  in (2.11), for whose elements  $q$  the right  $L$ -linearity condition  $q(x)(cl) = q(x)(c)l$  holds, for  $x \in \Sigma$ ,  $c \in \mathcal{C}$  and  $l \in L$ . The connecting maps are expressed in terms of the connecting maps in (2.11) as

$$(2.16) \quad \begin{aligned} \blacklozenge : \widetilde{Q} \otimes_{{}_L\text{Hom}_L(\mathcal{D}, T)} {}_L\text{Hom}^{\mathcal{D}}(\mathcal{D}, \Sigma) &\rightarrow {}^{\mathcal{C}}\text{End}^{\mathcal{D}}(\mathcal{C})^{op}, & q \otimes p &\mapsto (c \mapsto c_{[0]}(q \blacktriangleright p(c_{[1]}))) \\ \diamond : {}_L\text{Hom}^{\mathcal{D}}(\mathcal{D}, \Sigma) \otimes_{{}^{\mathcal{C}}\text{End}^{\mathcal{D}}(\mathcal{C})^{op}} \widetilde{Q} &\rightarrow {}_L\text{Hom}_L(\mathcal{D}, T), & p \otimes q &\mapsto (d \mapsto p(d) \vee q), \end{aligned}$$

where a Sweedler type index notation  $c \mapsto c_{[0]} \otimes_L c_{[1]}$  is used for the  $\mathcal{D}$ -coaction in  $\mathcal{C}$  (implicit summation is understood). For an explanation of the categorical origin of this Morita context and explicit form of the bimodule structures we refer to [5, Proposition 3.1] and its corrigendum. In order to generalize in Theorem 2.7 below some of the claims in Theorem 3.6 and Theorem 4.1 in [5] beyond the case when the connecting map  $\blacklozenge$  is surjective (in particular when  $\Sigma$  is a *cleft* bicomodule), we need the following

**Lemma 2.6.** *Let  $\mathcal{C}$  be a coring over a unital ring  $A$  and let  $R \subseteq {}^*\mathcal{C}$  be a (non-unital) subring. If  $\mathcal{C}$  is a firm right  $R$ -module and the left regular  $R$ -module is flat, then every right  $\mathcal{C}$ -comodule  $M$  is a firm right  $R$ -module, with action  $mr := m^{[0]}r(m^{[1]})$ .*

*Proof.* For any  $M \in \mathcal{M}^{\mathcal{C}}$ , there is a sequence of isomorphisms

$$M \otimes_R R \cong (M \otimes^{\mathcal{C}} \mathcal{C}) \otimes_R R \cong M \otimes^{\mathcal{C}} (\mathcal{C} \otimes_R R) \cong M \otimes^{\mathcal{C}} \mathcal{C} \cong M$$

mapping  $M \otimes_R R \ni m \otimes_R r$  to  $m^{[0]}r(m^{[1]})$ . The second isomorphism holds since  $R$  is a flat left  $R$ -module and the penultimate isomorphism holds since  $\mathcal{C}$  is a firm right  $R$ -module.  $\square$

The ring  ${}^{\mathcal{C}}\text{End}^{\mathcal{D}}(\mathcal{C})^{op}$  in the Morita context (2.15) is a (unital) subring of the ring  ${}^{\mathcal{C}}\text{End}(\mathcal{C})^{op} \cong {}^*\mathcal{C}$ . Hence any right  $\mathcal{C}$ -comodule  $N$  is a right  ${}^{\mathcal{C}}\text{End}^{\mathcal{D}}(\mathcal{C})^{op}$ -module via

$$(2.17) \quad nu := n^{[0]}\epsilon_{\mathcal{C}}(u(n^{[1]})), \quad \text{for } n \in N, u \in {}^{\mathcal{C}}\text{End}^{\mathcal{D}}(\mathcal{C})^{op}.$$

Obviously, the  $\mathcal{C}$ -coaction  $\rho^N : N \rightarrow N \otimes_A \mathcal{C}$  on  $N$  is a right  ${}^{\mathcal{C}}\text{End}^{\mathcal{D}}(\mathcal{C})^{op}$ -linear morphism, i.e.  $(nu)^{[0]} \otimes_A (nu)^{[1]} = n^{[0]} \otimes_A n^{[1]}u = n^{[0]} \otimes_A n^{[1]}\epsilon(u(n^{[2]}))$ .

**Theorem 2.7.** *Let  $\mathcal{D}$  be a coring over a unital ring  $L$ , which is a pure right extension of a coring  $\mathcal{C}$  over a unital ring  $A$ . Let  $\Sigma$  be an  $L$ - $\mathcal{C}$  bicomodule and consider the associated Morita context (2.15). Let  $R$  be a firm ring and a left ideal in  $\tilde{Q} \blacklozenge_L \text{Hom}^{\mathcal{D}}(\mathcal{D}, \Sigma)$ , such that the left regular  $R$ -module is flat and  $\mathcal{C}$  is a firm right  $R$ -module. Then*

$$(2.18) \quad \text{can} : \text{Hom}_A(\Sigma, -) \otimes_T \Sigma \rightarrow - \otimes_A \mathcal{C}, \quad \text{can}_N(\phi_N \otimes_T x) = \phi_N(x^{[0]}) \otimes_A x^{[1]}$$

*is a natural isomorphism and the functor  $\text{Hom}^{\mathcal{C}}(\Sigma, -) : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_T$  is fully faithful.*

*Proof.* By Corollary 1.21 there exists an  $R$ -bimodule map

$$R \rightarrow \tilde{Q} \otimes_{L\text{Hom}_L(\mathcal{D}, T)} {}_L\text{Hom}^{\mathcal{D}}(\mathcal{D}, \Sigma), \quad r \mapsto \tilde{j}_r \otimes j_r$$

(with implicit summation understood), such that  $\tilde{j}_r \blacklozenge j_r = r$ . We claim that the inverse of (2.18) is given by the well defined map

$$\text{can}_N^{-1}(n \otimes_A cr) = n\tilde{j}_r(c_{[0]})(-) \otimes_T j_r(c_{[1]}), \quad \text{for } n \in N, c \in \mathcal{C}, r \in R.$$

Indeed, the same arguments used to prove [5, Theorem 3.6] yield

$$(2.19) \quad (\text{can}_N \circ \text{can}_N^{-1})(n \otimes_A cr) = n \otimes_A cr \quad \text{and} \quad (\text{can}_N^{-1} \circ \text{can}_N)(\phi_N \otimes_T xr) = \phi_N \otimes_T xr,$$

for  $n \in N$ ,  $c \in \mathcal{C}$ ,  $\phi_N \in \text{Hom}_A(\Sigma, N)$ ,  $x \in \Sigma$  and  $r \in R$ , where the  $R$ -actions are induced by (2.17).  $\mathcal{C}$  is a firm right  $R$ -module by assumption.  $R$  is a non-unital subring in  ${}^{\mathcal{C}}\text{End}^{\mathcal{D}}(\mathcal{C})^{op} \subseteq {}^{\mathcal{C}}\text{End}(\mathcal{C})^{op} \cong {}^*\mathcal{C}$ , hence  $\Sigma$  is a firm right  $R$ -module by Lemma 2.6. Thus  $N \otimes_A \mathcal{C}R = N \otimes_A \mathcal{C}$  and  $\text{Hom}_A(\Sigma, N) \otimes_T \Sigma R = \text{Hom}_A(\Sigma, N) \otimes_T \Sigma$ . Therefore (2.19) proves that (2.18) is a natural isomorphism.

In view of [5, Lemma 2.1 (2)], for any right  $\mathcal{C}$ -comodule  $N$  with coaction  $\rho^N$  and  $n \in N$ ,  $r \in R$ ,  $(\text{can}_N^{-1} \circ \rho^N)(nr) \in \text{Hom}^{\mathcal{C}}(\Sigma, N) \otimes_T \Sigma$ . Since  $N$  is a firm right  $R$ -module Lemma by 2.6, this shows that the range of  $\text{can}_N^{-1} \circ \rho^N$  lies within  $\text{Hom}^{\mathcal{C}}(\Sigma, N) \otimes_T \Sigma$ . The same computations in [5, Theorem 4.1] yield that corestriction of  $\text{can}_N^{-1} \circ \rho^N$  gives the inverse of (2.3), hence  $\text{Hom}^{\mathcal{C}}(\Sigma, -) : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_T$  is a fully faithful functor.  $\square$

**2.3. A Morita context connecting two comodules.** Two objects  $\Sigma$  and  $\Lambda$  in a  $k$ -linear category determine a Morita context

$$(2.20) \quad \mathbb{M}(\Sigma, \Lambda) = (\text{End}(\Sigma), \text{End}(\Lambda), \text{Hom}(\Lambda, \Sigma), \text{Hom}(\Sigma, \Lambda), \square, \blacksquare),$$

where multiplication, all bimodule structures and also the connecting maps are given by composition in the category (what will be denoted by juxtaposition throughout). In this section we study (reduction of) the Morita context (2.20), determined by two objects  $\Sigma$  and  $\Lambda$  in the  $k$ -linear category of right comodules of a coring  $\mathcal{C}$  over a unital ring  $A$  over a commutative ring  $k$ . Throughout the section let  $B \subseteq \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda)\text{Hom}^{\mathcal{C}}(\Lambda, \Sigma)$  be a left ideal and an idempotent ring and  $W := \text{Hom}^{\mathcal{C}}(\Lambda, \Sigma)B\text{Hom}^{\mathcal{C}}(\Sigma, \Lambda)$ .

We can consider the reduced form (1.8) (or equivalently, (1.9), see Remark 1.18 (iii)) of the Morita context (2.20), i.e.

$$(2.21) \quad (W, B, \text{Hom}^{\mathcal{C}}(\Lambda, \Sigma) \otimes_B B, B \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda), \overline{\square}, \overline{\blacksquare}).$$

We obtain the following (not necessarily commutative) diagram of adjoint functors. (In order to see that  $G_{\Sigma}$  and  $G_{\Lambda}$  are well defined, consult Theorem 1.1 (i).)

$$(2.22) \quad \begin{array}{ccc} & \mathcal{M}^{\mathcal{C}} & \\ \begin{array}{c} \swarrow G_{\Sigma} = \text{Hom}^{\mathcal{C}}(\Sigma, -)W \otimes_W W \\ \searrow F_{\Sigma} = - \otimes_W \Sigma \end{array} & & \begin{array}{c} \nwarrow G_{\Lambda} = \text{Hom}^{\mathcal{C}}(\Lambda, -)B \otimes_B B \\ \swarrow F_{\Lambda} = - \otimes_B \Lambda \end{array} \\ \mathcal{M}_{\widetilde{W}} \cong \mathcal{M}_W & \begin{array}{c} \xleftarrow{- \otimes_W \text{Hom}^{\mathcal{C}}(\Lambda, \Sigma) \otimes_B B} \\ \xrightarrow{- \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda)} \end{array} & \mathcal{M}_B \cong \mathcal{M}_{\widetilde{B}} \end{array}$$

where  $\widetilde{W} = W \otimes_W W$ , as before. If  $W$  is a firm ring (i.e.  $W = \widetilde{W}$ ) then the adjunction  $(F_{\Sigma}, G_{\Sigma})$  reduces to (2.8). If we consider  $F_{\Sigma}$  and  $G_{\Sigma}$  as functors between  $\mathcal{M}^{\mathcal{C}}$  and  $\mathcal{M}_W$ , the unit and counit are defined as

$$\begin{aligned} \nu_N^{\Sigma} : N &\rightarrow \text{Hom}^{\mathcal{C}}(\Sigma, N \otimes_W \Sigma)W \otimes_W W, & n &\mapsto ((n^w)^{w'} \otimes_W -)w' \otimes_W w, \\ \zeta_M^{\Sigma} : \text{Hom}^{\mathcal{C}}(\Sigma, M)W \otimes_W W \otimes_W \Sigma &\rightarrow M, & \phi w \otimes_W w' \otimes_W x &\mapsto \phi(ww'x), \end{aligned}$$

for  $M \in \mathcal{M}^{\mathcal{C}}$  and  $N \in \mathcal{M}_W$ . Similarly, we define the unit  $\nu^{\Lambda}$  and the counit  $\zeta^{\Lambda}$  for the adjunction  $(F_{\Lambda}, G_{\Lambda})$ .

The aim of Proposition 2.8 is to relate the functors  $G_{\Sigma} : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_W$  and  $G_{\Lambda} : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_B$ , i.e. to show that the outer triangle in diagram (2.22) is commutative up to a natural isomorphism.

**Proposition 2.8.** *For a unital ring  $A$ , let  $\Sigma$  and  $\Lambda$  be right comodules of an  $A$ -coring  $\mathcal{C}$  and  $B \subseteq \text{End}^{\mathcal{C}}(\Lambda)$  and  $W \subseteq \text{End}^{\mathcal{C}}(\Sigma)$  as above. Then, for any right  $\mathcal{C}$ -comodule  $M$ , there is a right  $W$ -module isomorphism*

$$\text{Hom}^{\mathcal{C}}(\Sigma, M)W \otimes_W W \cong \text{Hom}^{\mathcal{C}}(\Lambda, M)B \otimes_B B \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda).$$

*Proof.* Consider the  $B$ -reduced form (2.21) of the Morita context (2.20). The right  $W$ -module  $\text{Hom}^{\mathcal{C}}(\Sigma, M)W \otimes_W W$  and the right  $B$ -module  $\text{Hom}^{\mathcal{C}}(\Lambda, M)B \otimes_B B$  are firm by Theorem 1.1 (i), for any right  $\mathcal{C}$ -comodule  $M$ . Therefore, by Lemma 1.10, the morphisms  $\text{Hom}^{\mathcal{C}}(\Sigma, M)W \otimes_W W \otimes_W \overline{\square}$  and  $\text{Hom}^{\mathcal{C}}(\Lambda, M)B \otimes_B B \otimes_B \overline{\blacksquare}$  are isomorphisms. Furthermore, composition of  $\mathcal{C}$ -comodule morphisms defines maps, for any

$$M \in \mathcal{M}^{\mathcal{C}},$$

$$\begin{aligned}\omega_1 &: \text{Hom}^{\mathcal{C}}(\Sigma, M)W \otimes_W W \otimes_W \text{Hom}^{\mathcal{C}}(\Lambda, \Sigma) \otimes_B B \rightarrow \text{Hom}^{\mathcal{C}}(\Lambda, M)B, \\ \omega_2 &: \text{Hom}^{\mathcal{C}}(\Lambda, M)B \otimes_B B \otimes_B B \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda) \rightarrow \text{Hom}^{\mathcal{C}}(\Lambda, M)B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda), \\ \omega_3 &: \text{Hom}^{\mathcal{C}}(\Lambda, \Sigma) \otimes_B B \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda) \rightarrow W.\end{aligned}$$

Obviously,  $\omega_1$  is a right  $B$ -module map,  $\omega_2$  is right  $W$ -linear and  $\omega_3$  is  $W$ - $W$  bilinear. By Lemma 1.14 (ii),  $\text{Hom}^{\mathcal{C}}(\Lambda, M)B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda) = \text{Hom}^{\mathcal{C}}(\Lambda, M)B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda)W$  is a right  $W$ -submodule of  $\text{Hom}^{\mathcal{C}}(\Sigma, M)W$ . Hence there is a well defined map

$$\begin{aligned}(\omega_2 \otimes_W \omega_3) &\circ ((\text{Hom}^{\mathcal{C}}(\Lambda, M)B \otimes_B B \otimes_B \overline{\blacksquare})^{-1} \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda)) \\ &\circ ((\text{Hom}^{\mathcal{C}}(\Lambda, M)B \otimes_B \mu_B)^{-1} \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda))\end{aligned}$$

from  $\text{Hom}^{\mathcal{C}}(\Lambda, M)B \otimes_B B \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda)$  to  $\text{Hom}^{\mathcal{C}}(\Sigma, M)W \otimes_W W$ . A routine computation shows that it is an isomorphism with inverse

$$(\omega_1 \otimes_B B \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda)) \circ (\text{Hom}^{\mathcal{C}}(\Sigma, M)W \otimes_W W \otimes_W \overline{\square})^{-1} \circ (\text{Hom}^{\mathcal{C}}(\Sigma, M)W \otimes_W \mu_W)^{-1}.$$

This ends the proof.  $\square$

**Corollary 2.9.** *Let  $\Sigma$  and  $\Lambda$  be right comodules of a coring  $\mathcal{C}$  over a unital ring  $A$ , and let  $B \subseteq \text{End}^{\mathcal{C}}(\Lambda)$  and  $W \subseteq \text{End}^{\mathcal{C}}(\Sigma)$  be non-unital subrings as in Proposition 2.8. Then the following assertions hold.*

- (i) *If  $\zeta_{\Sigma}^{\Lambda}$  or  $\zeta_{\Lambda}^{\Sigma}$  is an isomorphism, then the functor  $F_{\Sigma} : \mathcal{M}_W \rightarrow \mathcal{M}^{\mathcal{C}}$  is fully faithful if and only if  $F_{\Lambda} : \mathcal{M}_B \rightarrow \mathcal{M}^{\mathcal{C}}$  is fully faithful;*
- (ii) *The functor  $G_{\Sigma} : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_W$  is fully faithful if and only if  $G_{\Lambda} : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_B$  is fully faithful;*
- (iii) *The functor  $G_{\Sigma} : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_W$  is an equivalence if and only if  $G_{\Lambda} : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_B$  is an equivalence.*

*Proof.* (i). For any  $N \in \mathcal{M}_W$ , there is a natural morphism

$$N \otimes_W \zeta_{\Sigma}^{\Lambda} : N \otimes_W \text{Hom}^{\mathcal{C}}(\Lambda, \Sigma) \otimes_B B \otimes_B \Lambda \rightarrow N \otimes_W \Sigma.$$

Therefore,  $F_{\Sigma}$  is naturally isomorphic to the composite of the functors  $F_{\Lambda}$  and  $- \otimes_W \text{Hom}^{\mathcal{C}}(\Lambda, \Sigma) \otimes_B B$ , provided  $\zeta_{\Sigma}^{\Lambda}$  is an isomorphism. Since we know that  $- \otimes_W \text{Hom}^{\mathcal{C}}(\Lambda, \Sigma) \otimes_B B : \mathcal{M}_W \rightarrow \mathcal{M}_B$  is an equivalence (see Remark 1.18 applied to the  $B$ -reduced form of the Morita context (2.20)), this proves the claim.

(ii)& (iii). By Proposition 2.8,  $G_{\Sigma}$  is naturally isomorphic to the composite of  $G_{\Lambda}$  and the equivalence functor  $- \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, \Lambda) : \mathcal{M}_B \rightarrow \mathcal{M}_W$ , which proves both claims.  $\square$

In [5, Proposition 2.7] we proved that, in the case when  $\mathcal{C}$  is a finitely generated projective left  $A$ -module, the Morita context  $\mathbb{M}(\Sigma)$  in (2.11) is strict if and only if the Strong Structure Theorem holds, that is,  $\text{Hom}^{\mathcal{C}}(\Sigma, -) : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_T$  is an equivalence. The aim of the rest of current section is to extend this result beyond the case when  $\mathcal{C}$  is a finitely generated projective left  $A$ -module.

In order to apply the results of this section, in addition to  $\Sigma$  we need a second  $\mathcal{C}$ -comodule. In what follows we give sufficient and necessary conditions under which the range  $B$  of the connecting map  $\blacktriangledown$  in the Morita context (2.11) has a  $B$ - $\mathcal{C}$  bicomodule structure such that the corresponding adjunction  $(F_B, G_B)$  (see (2.22)) is an

equivalence. In the case when these conditions hold, we apply Corollary 2.9 to prove that also the adjunction  $(F_\Sigma, G_\Sigma)$  is an equivalence.

Recall (e.g. from [30] or [29]) that for a left module  $P$  over a unital ring  $A$ , the *finite topology* on  ${}^*P := {}_A\text{Hom}(P, A)$  is generated by the open sets  $\mathcal{O}(f, p_1, \dots, p_n) = \{ g \in {}^*P \mid g(p_i) = f(p_i), i = 1, \dots, n \}$ . The left  $A$ -module  $P$  is said to be *weakly locally projective* if every finitely generated submodule of  $P$  has a dual basis in  $P \times {}^*P$ . Equivalently, if and only if  ${}^*P$  satisfies the  $\alpha$ -condition, meaning that the map

$$M \otimes_A P \rightarrow \text{Hom}_{Ab}({}^*P, M), \quad m \otimes_A p \mapsto (f \mapsto mf(p))$$

is injective, for every right  $A$ -module  $M$ . A non-unital ring  $B$  has right *local units* if for any finite subset  $\{b_1, \dots, b_n\}$  of  $B$  there exists an element  $e \in B$  such that  $b_i e = b_i$ , for all  $i = 1, \dots, n$ . If  $B$  is a ring with right local units then it is in particular firm and its left regular module is flat.

**Lemma 2.10.** *For a unital ring  $A$ , let  $\Sigma$  be a right comodule of an  $A$ -coring  $\mathcal{C}$  and let  $B := Q \blacktriangledown \Sigma$  be the range of the connecting map  $\blacktriangledown$  in the Morita context  $\mathbb{M}(\Sigma)$  in (2.11). Assume that the left regular  $B$ -module extends to a  $B$ - $\mathcal{C}$  bicomodule such that the connecting map  $\blacktriangledown$  corestricts to a  $B$ - $\mathcal{C}$  bicomodule epimorphism  $Q \otimes_T \Sigma \rightarrow B$ . Then  $F_B$  has a left inverse, the functor  $\tilde{F}_B$ , sending a right  $\mathcal{C}$ -comodule  $M$  to the right  $B$ -module  $M$ , with action  $mb := m^{[0]}b(m^{[1]})$ , and acting on the morphisms as the identity map.*

*Proof.* For  $q \in Q$ ,  $x \in \Sigma$ ,  $b \in B$  and  $c \in \mathcal{C}$ ,

$$\begin{aligned} (q(x)^{[0]}b(q(x)^{[1]}))(c) &= q(x^{[0]})(c)b(x^{[1]}) = b(q(x^{[0]})(c)x^{[1]}) \\ &= b(c^{(1)}q(x)(c^{(2)})) = (q(x)b)(c). \end{aligned}$$

The first equality follows by the right  $\mathcal{C}$ -colinearity of  $\blacktriangledown : q \otimes x \mapsto q(x)$  and the form of the right  $A$ -action on  ${}^*\mathcal{C}$ . The second equality follows by the left  $A$ -linearity of  $b \in B \subseteq {}^*\mathcal{C}$ . The penultimate equality follows by the defining property of  $q \in Q$  while the last one follows by the form of the multiplication in  ${}^*\mathcal{C}$ . Since  $B$  is the range of  $\blacktriangledown$ , we conclude that  $b^{[0]}b'(b^{[1]}) = bb'$ , for all  $b, b' \in B$ . Thus  $\tilde{F}_B \circ F_B$  takes a firm right  $B$ -module  $N$  to the right  $B$ -module  $N$ , with action

$$n \otimes b' \mapsto n^b b^{[0]}b'(b^{[1]}) = n^b b b' = n b',$$

where  $\mathbf{d}_{N,B}(n) = n^b \otimes_B b$  is the unique element of  $N \otimes_B B$  such that  $n^b b = n$ . This proves  $\tilde{F}_B \circ F_B = \mathcal{M}_B$ .  $\square$

**Theorem 2.11.** *For a unital ring  $A$ , let  $\Sigma$  be a right comodule of an  $A$ -coring  $\mathcal{C}$  and let  $B := Q \blacktriangledown \Sigma$  be the range of the connecting map  $\blacktriangledown$  in the Morita context  $\mathbb{M}(\Sigma)$  in (2.11). The following assertions are equivalent.*

- (i) *The left regular  $B$ -module extends to a  $B$ - $\mathcal{C}$  bicomodule such that the connecting map  $\blacktriangledown$  corestricts to a  $B$ - $\mathcal{C}$  bicomodule epimorphism  $Q \otimes_T \Sigma \rightarrow B$ , and  $F_B : \mathcal{M}_B \rightarrow \mathcal{M}^{\mathcal{C}}$  is an isomorphism;*
- (ii)  *$\mathcal{C}$  is weakly locally projective as a left  $A$ -module and  $B$  is dense in the finite topology on  ${}^*\mathcal{C}$ ;*
- (iii)  *$\mathcal{C}$  is weakly locally projective as a left  $A$ -module,  $B$  has right local units (in particular,  $B$  is a firm ring) and  $\mathcal{C}$  is firm as a right  $B$ -module;*



(iv) *The left regular  $B$ -module extends to a  $B$ - $\mathcal{C}$  bicomodule such that the connecting map  $\blacktriangledown$  corestricts to a  $B$ - $\mathcal{C}$  bicomodule epimorphism  $Q \otimes_T \Sigma \rightarrow B$ ,  $\mathcal{C}$  is firm as a right  $B$ -module and the left regular  $B$ -module is flat.*

*Proof.* (i) $\Rightarrow$ (ii). By Lemma 2.10,  $F_B$  has a left inverse, the functor  $\tilde{F}_B$ , sending a right  $\mathcal{C}$ -comodule  $M$  to the right  $B$ -module  $M$ , with action  $mb := m^{[0]}b(m^{[1]})$ , and acting on the morphisms as the identity map. Under assumptions (i),  $F_B$  is an isomorphism. Hence  $\tilde{F}_B = F_B^{-1}$ . The category of firm modules over an idempotent ring was proven to be a Grothendieck category by Marín in [24]. Since  $F_B^{-1}(\mathcal{C}) = \mathcal{C}$  is a firm right  $B$ -module (with  $B$ -action  $cb = c^{(1)}b(c^{(2)})$ ), the smallest Grothendieck subcategory  $\sigma[\mathcal{C}_B]$  of  $\tilde{\mathcal{M}}_B$ , which contains  $\mathcal{C}$ , is contained in  $\mathcal{M}_B$ . On the other hand, by [29, Corollary 3.30], any right  $\mathcal{C}$ -comodule is subgenerated by  $\mathcal{C}$  as a right  $B$ -module, i.e.  $\mathcal{M}^{\mathcal{C}}$  is contained in  $\sigma[\mathcal{C}_B]$ . Hence the isomorphism  $\mathcal{M}^{\mathcal{C}} \cong \mathcal{M}_B$  implies  $\sigma[\mathcal{C}_B] \cong \mathcal{M}^{\mathcal{C}}$ . One can easily adapt the proof of [30, Theorem 3.5 (a) $\Rightarrow$ (d)] to conclude that the map

$$\alpha_{P,B} : P \otimes_A \mathcal{C} \rightarrow \text{Hom}_{Ab}(B, P), \quad p \otimes_A c \mapsto (b \mapsto pb(c))$$

is injective, for any right  $A$ -module  $P$ . This is equivalent to assertion (ii) by [29, Theorem 2.58].

(ii) $\Leftrightarrow$ (iii). This equivalence is proven in [29, Corollary 2.48].

(iii) $\Rightarrow$ (iv). Since  $B$  has right local units, its left regular module is flat. The existence of the required  $B$ - $\mathcal{C}$  bicomodule structure on  $B$  follows by a rationality argument. By construction, the map  $\blacktriangledown : Q \otimes_T \Sigma \rightarrow B$  is a surjective  $B$ - ${}^*\mathcal{C}$  bilinear map. Since  $Q \otimes_T \Sigma$  is a  $B$ - $\mathcal{C}$  bicomodule, it is in particular a rational right  ${}^*\mathcal{C}$ -module. Hence,  $B$  being the image of the map  $\blacktriangledown$ , it is a quotient of the rational  ${}^*\mathcal{C}$ -module  $Q \otimes_T \Sigma$ , and hence  $B$  itself is rational by [13, Proposition 4.2]. Therefore  $B$  is a  $B$ - $\mathcal{C}$  bicomodule and  $\blacktriangledown$  corestricts to a  $B$ - $\mathcal{C}$  bicomodule map.

(iv) $\Rightarrow$ (i). By Lemma 2.10,  $F_B$  has a left inverse  $\tilde{F}_B$ . Composition  $F_B \circ \tilde{F}_B$  makes sense by Lemma 2.6. The proof is completed by computing the coaction on the right  $\mathcal{C}$ -comodule  $F_B \circ \tilde{F}_B(M)$ , for  $M \in \mathcal{M}^{\mathcal{C}}$ . For  $q \in Q$ ,  $x \in \Sigma$  and  $c \in \mathcal{C}$ ,

$$q(x)^{[0]}(c)q(x)^{[1]} = q(x^{[0]})(c)x^{[1]} = c^{(1)}q(x)(c^{(2)}).$$

The first equality follows by right  $\mathcal{C}$ -colinearity of  $\blacktriangledown$ , and the second equality follows by the defining property of  $q \in Q$ . Hence, for  $b \in B$  and  $c \in \mathcal{C}$ ,  $b^{[0]}(c)b^{[1]} = c^{(1)}b(c^{(2)})$ . Note that by Lemma 2.6  $\tilde{F}_B(M) = \tilde{F}_B(M)B$ . The right  $\mathcal{C}$ -coaction on  $F_B \circ \tilde{F}_B(M)$  comes out as

$$\begin{aligned} mb &\mapsto m^{[0]}b^{[0]}(m^{[1]}) \otimes_A b^{[1]} = m^{[0]} \otimes_A b^{[0]}(m^{[1]})b^{[1]} = m^{[0]} \otimes_A m^{[1]}b(m^{[2]}) = \\ &(m^{[0]}b(m^{[1]}))^{[0]} \otimes_A (m^{[0]}b(m^{[1]}))^{[1]} = (mb)^{[0]} \otimes_A (mb)^{[1]}. \end{aligned}$$

This proves  $F_B \circ \tilde{F}_B = \mathcal{M}^{\mathcal{C}}$  hence the theorem.  $\square$

For a unital ring  $A$ , let  $\Sigma$  be a right comodule of an  $A$ -coring  $\mathcal{C}$  and let  $B := Q \blacktriangledown \Sigma$  be the range of the connecting map  $\blacktriangledown$  in the Morita context  $\mathbb{M}(\Sigma)$  in (2.11). Assume that the equivalent conditions in Theorem 2.11 hold. Then  $B$  is a firm ring, and we can consider the  $B$ -reduced form of  $\mathbb{M}(\Sigma)$

$$(2.23) \quad ((\Sigma \nabla Q)(\Sigma \nabla Q), B, \Sigma \otimes_B B, B \otimes_B Q, \tilde{\nabla}, \tilde{\blacktriangledown}),$$

where the connecting maps are given, for  $b, \tilde{b} \in B$ ,  $x \in \Sigma$ ,  $q \in Q$ , by

$$(2.24) \quad (x \otimes_B \tilde{b}) \tilde{\nabla} (b \otimes_B q) := x \tilde{b} \nabla bq \quad \text{and} \quad (b \otimes_B q) \tilde{\nabla} (x \otimes_B \tilde{b}) := bq \nabla x \tilde{b}.$$

On the other hand, under the conditions in Theorem 2.11,  $B$  is also a right  $\mathcal{C}$ -comodule, hence we can consider a Morita context  $\mathbb{M}(\Sigma, B)$  as in (2.20). In the next lemma we show that also the Morita context  $\mathbb{M}(\Sigma, B)$  admits a  $B$ -reduced form.

**Lemma 2.12.** *For a unital ring  $A$ , let  $\Sigma$  be a right comodule of an  $A$ -coring  $\mathcal{C}$  and let  $B := Q \nabla \Sigma$  be the range of the connecting map  $\nabla$  in the Morita context  $\mathbb{M}(\Sigma)$  in (2.11). Assume that the equivalent conditions in Theorem 2.11 hold, hence  $B$  is a  $B$ - $\mathcal{C}$  bicomodule. Then  $B$  is a left ideal in  $\text{Hom}^{\mathcal{C}}(\Sigma, B) \text{Hom}^{\mathcal{C}}(B, \Sigma)$ .*

*Proof.* Note first that there is a well-defined map

$$(2.25) \quad \gamma : \Sigma \rightarrow \text{Hom}^{\mathcal{C}}(B, \Sigma), \quad \gamma(y)(b) = yb.$$

That is, for all  $y \in \Sigma$ , the map  $\gamma(y) : B \rightarrow \Sigma$ ,  $\gamma(y)(b) = yb = y^{[0]}b(y^{[1]})$  is right  $\mathcal{C}$ -colinear. Indeed, for  $q \in Q$  and  $x \in \Sigma$ ,

$$\begin{aligned} \gamma(y)(q(x)^{[0]}) \otimes_A q(x)^{[1]} &= yq(x)^{[0]} \otimes_A q(x)^{[1]} = y^{[0]} \otimes_A q(x)^{[0]}(y^{[1]})q(x)^{[1]} \\ &= y^{[0]} \otimes_A q(x)^{[0]}(y^{[1]})x^{[1]} = y^{[0]} \otimes_A y^{[1]}q(x)(y^{[2]}) \\ &= (y^{[0]}q(x)(y^{[1]}))^{[0]} \otimes_A (y^{[0]}q(x)(y^{[1]}))^{[1]} \\ &= \gamma(y)(q(x))^{[0]} \otimes_A \gamma(y)(q(x))^{[1]}. \end{aligned}$$

The second and the last equalities follow by the form of the  $B$ -action on  $\Sigma$ . The third equality follows by the right  $\mathcal{C}$ -colinearity of  $\nabla$  and the fourth equality is a consequence of the defining property of  $q \in Q$ . The penultimate equality follows by right  $A$ -linearity of the  $\mathcal{C}$ -coaction on  $\Sigma$ .

Next, since any morphism in  $\text{End}^{\mathcal{C}}(B)$  is right  $B$ -linear, the map  $\beta : B \rightarrow \text{End}^{\mathcal{C}}(B)$ ,  $\beta(b)(b') = bb'$  turns  $B$  into a left ideal in  $\text{End}^{\mathcal{C}}(B)$ . Remark furthermore that  $\text{Hom}^{\mathcal{C}}(\Sigma, B) \text{Hom}^{\mathcal{C}}(B, \Sigma)$  is in a natural way a (two-sided) ideal in  $\text{End}^{\mathcal{C}}(B)$ . Finally, for any  $q \nabla x \in B$ , by right  $B$ -linearity of  $q \in Q$ , we have that  $\beta(q \nabla x) = q\gamma(x) \in \text{Hom}^{\mathcal{C}}(\Sigma, B) \text{Hom}^{\mathcal{C}}(B, \Sigma)$ . Hence the image of  $\beta$  and therefore  $B$  is a left ideal in  $\text{Hom}^{\mathcal{C}}(\Sigma, B) \text{Hom}^{\mathcal{C}}(B, \Sigma)$ .  $\square$

In view of Lemma 2.12, we can apply the theory developed at the beginning of this section to obtain a  $B$ -reduced form of the Morita context  $\mathbb{M}(\Sigma, B)$ , as in (2.21). Note that since  $B$  is a firm ring,  $M \otimes_B B \cong MB \otimes_B B$  and  $B \otimes_B N \cong B \otimes_B BN$ , for any  $M \in \widetilde{\mathcal{M}}_B$  and  $N \in {}_B \widetilde{\mathcal{M}}$ . Thus from (2.21) we obtain

$$(2.26) \quad (\text{Hom}^{\mathcal{C}}(B, \Sigma) B \text{Hom}^{\mathcal{C}}(\Sigma, B), B, \text{Hom}^{\mathcal{C}}(B, \Sigma) \otimes_B B, B \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, B), \tilde{\diamond}, \tilde{\blacklozenge}),$$

where the connecting maps are given, for  $b, \tilde{b} \in B$ ,  $\zeta \in \text{Hom}^{\mathcal{C}}(\Sigma, B)$  and  $\xi \in \text{Hom}^{\mathcal{C}}(B, \Sigma)$ , by

$$(2.27) \quad (\xi \otimes_B b) \tilde{\diamond} (\tilde{b} \otimes_B \zeta) := \xi \tilde{b} \zeta \quad \text{and} \quad (\tilde{b} \otimes_B \zeta) \tilde{\blacklozenge} (\xi \otimes_B b) := \tilde{b} \zeta \xi b.$$

**Proposition 2.13.** *For a unital ring  $A$ , let  $\Sigma$  be a right comodule of an  $A$ -coring  $\mathcal{C}$  and let  $B := Q \nabla \Sigma$  be the range of the connecting map  $\nabla$  in the Morita context  $\mathbb{M}(\Sigma)$  in (2.11). Assume that the equivalent conditions in Theorem 2.11 hold, hence  $B$  is a  $B$ - $\mathcal{C}$  bicomodule. Then the Morita contexts (2.23) and (2.26) are isomorphic.*

*Proof.* In terms of the map  $\gamma$  in (2.25), put

$$\alpha : \Sigma \otimes_B B \rightarrow \text{Hom}^{\mathcal{C}}(B, \Sigma) \otimes_B B, \quad x \otimes_B b \mapsto \gamma(x) \otimes_B b.$$

It is an isomorphism with inverse

$$\alpha^{-1} : \text{Hom}^{\mathcal{C}}(B, \Sigma) \otimes_B B \rightarrow \Sigma \otimes_B B, \quad \xi \otimes_B bb' \mapsto \xi(b) \otimes_B b'.$$

Indeed,

$$\begin{aligned} (\alpha^{-1} \circ \alpha)(x \otimes_B bb') &= \gamma(x)(b) \otimes_B b' = xb \otimes_B b' = x \otimes_B bb; \\ (\alpha \circ \alpha^{-1})(\xi \otimes_B bb') &= \gamma(\xi(b)) \otimes_B b' = \xi b \otimes_B b' = \xi \otimes_B bb', \end{aligned}$$

where in the penultimate equality of the second computation we used that a right  $\mathcal{C}$ -comodule map  $\xi$  is a right module map for  $B \subseteq {}^*\mathcal{C}$ , hence for all  $b' \in B$ ,  $\gamma(\xi(b))(b') = \xi(b)b' = \xi(bb') = (\xi b)(b')$ .

By right  $\mathcal{C}$ -colinearity of  $\blacktriangledown$ ,  $Q \subseteq \text{Hom}^{\mathcal{C}}(\Sigma, B)$ . Conversely, for any  $\zeta \in \text{Hom}^{\mathcal{C}}(\Sigma, B)$  and  $y \in \Sigma$ , there exist (non-unique) elements  $q_i \in Q$  and  $x_i \in \Sigma$  such that  $B \ni \zeta(y) = \sum_i q_i(x_i)$ . Thus, for  $c \in \mathcal{C}$ ,

$$\begin{aligned} \zeta(y^{[0]})(c)y^{[1]} &= \zeta(y)^{[0]}(c)\zeta(y)^{[1]} = \sum_i q_i(x_i)^{[0]}(c)q_i(x_i)^{[1]} \\ &= \sum_i q_i(x_i^{[0]})(c)x_i^{[1]} = \sum_i c^{(1)}q_i(x_i)(c^{(2)}) = c^{(1)}\zeta(y)(c^{(2)}). \end{aligned}$$

The first equality follows by the  $\mathcal{C}$ -colinearity of  $\zeta \in \text{Hom}^{\mathcal{C}}(\Sigma, B)$ . The third equality follows by the  $\mathcal{C}$ -colinearity of  $q_i$ , for all values of the index  $i$ . In order to conclude the penultimate equality we used the defining property of  $q_i \in Q$ , for any index  $i$ . This proves that  $\text{Hom}^{\mathcal{C}}(\Sigma, B) \subseteq Q$ , hence the obvious map

$$\beta : B \otimes_B Q \rightarrow B \otimes_B \text{Hom}^{\mathcal{C}}(\Sigma, B), \quad b \otimes_B q \mapsto b \otimes_B q$$

establishes an isomorphism. One checks easily that the isomorphisms  $\alpha$  and  $\beta$  are compatible with the connecting maps (2.24) and (2.27). Thus in particular the ranges of the connecting maps (2.24) and (2.27) are coinciding (non-unital) subrings of the endomorphism rings  $\text{End}^{\mathcal{C}}(B)$  and  $T$ , respectively. That is,  $(\Sigma \blacktriangledown Q)(\Sigma \blacktriangledown Q) = \text{Hom}^{\mathcal{C}}(B, \Sigma)B\text{Hom}^{\mathcal{C}}(\Sigma, B)$ . The proof is completed by checking the bimodule map properties of  $\alpha$  and  $\beta$ , what is left to the reader.  $\square$

The following theorem generalizes [29, Theorem 5.22] and hence [13, Theorem 5.3] and [11, Theorem 4.15] beyond the case when  $\blacktriangledown$  is surjective.

**Theorem 2.14.** *For a unital ring  $A$ , let  $\Sigma$  be a right comodule of an  $A$ -coring  $\mathcal{C}$  and consider the Morita context  $\mathbb{M}(\Sigma)$  in (2.11). Put  $B := Q \blacktriangledown \Sigma$  and  $W := (\Sigma \blacktriangledown Q)(\Sigma \blacktriangledown Q)$ . Assume that the equivalent conditions in Theorem 2.11 hold. Then*

$$(2.28) \quad G_{\Sigma} := \text{Hom}^{\mathcal{C}}(\Sigma, -)W \otimes_W W : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_W$$

*is an equivalence.*

*Proof.* Since  $F_B$  is an equivalence functor by assumption, so is its adjoint  $G_B$ . By Lemma 2.12  $B$  is a left ideal in  $\text{Hom}^{\mathcal{C}}(\Sigma, B)\text{Hom}^{\mathcal{C}}(B, \Sigma)$  and by assumption  $B$  is a firm ring. Thus the claim is an immediate consequence of Proposition 2.13 and Corollary 2.9.  $\square$

If the functor (2.28) is fully faithful (as e.g. in Theorem 2.14) then  $\Sigma$  is a generator in  $\mathcal{M}^c$ . Under the conditions in Theorem 2.14,  $\mathcal{C}$  is a flat left  $A$ -module (hence the forgetful functor  $\mathcal{M}^c \rightarrow \mathcal{M}_A$  preserves and reflects monomorphisms) and (2.28) (thus also  $F_\Sigma = - \otimes_W \Sigma : \mathcal{M}_W \rightarrow \mathcal{M}^c$ ) is an equivalence. Hence  $\Sigma$  is a faithfully flat left  $W$ -module. Moreover, the following proposition holds.

**Proposition 2.15.** *In the situation described in Theorem 2.14, the functor  $- \otimes_W \Sigma : \mathcal{M}_W \rightarrow \mathcal{M}_A$  has a right adjoint, the functor*

$$(2.29) \quad \text{Hom}_A(\Sigma, -)W \otimes_W W \simeq - \otimes_A \text{Hom}_A(\Sigma, A)W \otimes_W W.$$

Note that if  $W$  is a unital ring then the equivalence of the two forms of the functor in (2.29) is equivalent to finitely generated projectivity of  $\Sigma$  as a right  $A$ -module. If  $W$  is a firm ring and  $\Sigma$  is a firm left  $W$ -module, then this equivalence is equivalent to  $W$ -firm projectivity of the right  $A$ -module  $\Sigma$  by Theorem 1.20.

*Proof of Proposition 2.15.* The functor  $\text{Hom}_A(\Sigma, -)W \otimes_W W$  is equal to the composite of  $- \otimes_A \mathcal{C} : \mathcal{M}_A \rightarrow \mathcal{M}^c$  and the equivalence functor  $G_\Sigma : \mathcal{M}^c \rightarrow \mathcal{M}_W$  in (2.28). Since both of these functors possess as well a left adjoint as a right adjoint, also  $\text{Hom}_A(\Sigma, -)W \otimes_W W$  possesses both left and right adjoints. Furthermore, by Theorem 1.1 (vi), there are natural equivalences (of left adjoint functors)

$$\begin{aligned} \text{Hom}_A(\Sigma, -)W \otimes_W W &\simeq \text{Hom}_A(\Sigma, -) \otimes_{\widetilde{W}} \widetilde{W} && \text{and} \\ - \otimes_A \text{Hom}_A(\Sigma, A)W \otimes_W W &\cong - \otimes_A \text{Hom}_A(\Sigma, A) \otimes_{\widetilde{W}} \widetilde{W}, \end{aligned}$$

where  $\widetilde{W} = W \otimes_W W$  as before. By [28, Theorem 3.1 (i) $\Rightarrow$ (iii)] there is a natural equivalence

$$\text{Hom}_A(\Sigma, -) \otimes_{\widetilde{W}} \widetilde{W} \cong - \otimes_A \text{Hom}_A(\Sigma, A) \otimes_{\widetilde{W}} \widetilde{W}.$$

We conclude the claim by combining these isomorphisms.  $\square$

**Example 2.16.** Let  $\mathcal{C}$  be an  $A$ -coring that is locally projective as left  $A$ -module and put  $B = \text{Rat}(*\mathcal{C})$ , the rational part of the left dual of  $\mathcal{C}$ . In several situations  $B$  is dense in the finite topology of  $\mathcal{C}$ . E.g. if  $\mathcal{C}$  is a locally Frobenius coring as defined in [21]. If the base ring  $A$  is a PF-ring, then the definition of locally Frobenius coring is equivalent to the definition of a co-Frobenius coring. If  $A$  is a QF ring, then  $B$  is dense in the finite topology of  $*\mathcal{C}$  if and only if  $\mathcal{C}$  is a semiperfect coring [12].

Morita contexts of type (2.11) s.t.  $\blacktriangledown$  is surjective onto  $B = \text{Rat}(*\mathcal{C})$  have been considered extensively in e.g. [11], [13], [3] and fit into the framework of this section.

**2.4. Coseparable corings.** Recall that an  $A$ -coring  $\mathcal{C}$  is said to be *coseparable* if and only if there exists a  $\mathcal{C}$ -bilinear left inverse  $\mu : \mathcal{C} \otimes_A \mathcal{C} \rightarrow \mathcal{C}$  of the comultiplication  $\Delta$ . If we denote  $\gamma := \varepsilon \circ \mu : \mathcal{C} \otimes_A \mathcal{C} \rightarrow A$ , which is an  $A$ -bimodule map, then the following identities hold, for all  $c, d \in \mathcal{C}$ .

$$(2.30) \quad c^{(1)}\gamma(c^{(2)} \otimes_A d) = \mu(c \otimes_A d) = \gamma(c \otimes_A d^{(1)})d^{(2)}; \quad \gamma(c^{(1)} \otimes_A c^{(2)}) = \varepsilon(c).$$

The following theorem extends [8, Theorem 2.6 and Proposition 2.7].

**Proposition 2.17.** *Let  $\mathcal{C}$  be a coseparable coring over a unital ring  $A$ . Then  $\mathcal{C}$  is a firm ring. The categories  $\mathcal{M}^c$  and  $\mathcal{M}_{\mathcal{C}}$  are isomorphic, as are the categories  ${}^c\mathcal{M}$  and  ${}_c\mathcal{M}$ . Moreover, for all  $P \in \mathcal{M}^c$  and  $Q \in {}^c\mathcal{M}$ , the natural morphism*

$$P \otimes^{\mathcal{C}} Q \xrightarrow{\iota} P \otimes_A Q \xrightarrow{\pi} P \otimes_{\mathcal{C}} Q,$$

obtained by composing the canonical monomorphism  $\iota$  with the canonical epimorphism  $\pi$ , is an isomorphism with inverse

$$\beta : P \otimes_{\mathcal{C}} Q \rightarrow P \otimes^{\mathcal{C}} Q, \quad \beta(p \otimes_{\mathcal{C}} q) = p^{[0]} \otimes_A p^{[1]} \cdot q = p \cdot q^{[-1]} \otimes_A q^{[0]}.$$

*Proof.* Take any  $M \in \mathcal{M}^{\mathcal{C}}$  and define  $\mu_M : M \otimes_A \mathcal{C} \rightarrow M$  by the following composition.

$$\mu_M : M \otimes_A \mathcal{C} \xrightarrow{\rho^M \otimes_A \mathcal{C}} M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \xrightarrow{M \otimes_A \gamma} M \otimes_A A \cong M.$$

Remark that by (2.30),  $\mu_{\mathcal{C}} = \mu$ . Let us check that  $\mu_M$  is an associative action, i.e.  $\mu_M \circ (\mu_M \otimes_A \mathcal{C}) = \mu_M \circ (M \otimes_A \mu)$ , thus in particular  $\mu$  is an associative multiplication for  $\mathcal{C}$ . Indeed, for  $c, d \in \mathcal{C}$ ,

$$\begin{aligned} \mu_M(m^{[0]} \gamma(m^{[1]} \otimes_A c) \otimes_A d) &= m^{[0]} \gamma(m^{[1]} \gamma(m^{[2]} \otimes_A c) \otimes_A d) \\ &= m^{[0]} \gamma(\gamma(m^{[1]} \otimes_A c^{(1)}) c^{(2)} \otimes_A d) \\ &= m^{[0]} \gamma(m^{[1]} \otimes_A c^{(1)}) \gamma(c^{(2)} \otimes_A d) \\ &= m^{[0]} \gamma(m^{[1]} \otimes_A c^{(1)} \gamma(c^{(2)} \otimes_A d)) \\ &= m^{[0]} \gamma(m^{[1]} \otimes_A \mu(c \otimes_A d)). \end{aligned}$$

Next, let us prove that  $M$  is a firm  $\mathcal{C}$ -module, that is, the induced map  $\bar{\mu}_M : M \otimes_{\mathcal{C}} \mathcal{C} \rightarrow M$  is an isomorphism with inverse

$$\bar{\rho}^M : M \xrightarrow{\rho^M} M \otimes_A \mathcal{C} \xrightarrow{\pi} M \otimes_{\mathcal{C}} \mathcal{C}.$$

For all  $m \in M$ ,

$$\bar{\mu}_M \circ \bar{\rho}^M(m) = \bar{\mu}_M(m^{[0]} \otimes_{\mathcal{C}} m^{[1]}) = m^{[0]} \gamma(m^{[1]} \otimes_A m^{[2]}) = m^{[0]} \varepsilon(m^{[1]}) = m.$$

On the other hand, for all  $m \otimes_{\mathcal{C}} c \in M \otimes_{\mathcal{C}} \mathcal{C}$ ,

$$\begin{aligned} \bar{\rho}^M \circ \bar{\mu}_M(m \otimes_{\mathcal{C}} c) &= \bar{\rho}^M(m^{[0]} \gamma(m^{[1]} \otimes_A c)) = m^{[0]} \otimes_{\mathcal{C}} m^{[1]} \gamma(m^{[2]} \otimes_A c) \\ &= m^{[0]} \otimes_{\mathcal{C}} \mu(m^{[1]} \otimes_A c) = \mu_M(m^{[0]} \otimes_A m^{[1]}) \otimes_{\mathcal{C}} c \\ &= m^{[0]} \gamma(m^{[1]} \otimes_A m^{[2]}) \otimes_{\mathcal{C}} c = m \otimes_{\mathcal{C}} c. \end{aligned}$$

This defines a functor  $\Xi : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_{\mathcal{C}}$  acting on the morphisms as the identity. This justifies to denote from now on  $\mu_M(m \otimes_A c) = m \cdot c$ . Conversely, take  $M \in \mathcal{M}_{\mathcal{C}}$ . Since  $\mu$  is right  $\mathcal{C}$ -colinear,  $\Delta_{\mathcal{C}}$  is left  $\mathcal{C}$ -linear. Hence we can define  $\rho^M : M \rightarrow M \otimes_A \mathcal{C}$  as

$$\rho^M : M \xrightarrow{\bar{\mu}_M^{-1}} M \otimes_{\mathcal{C}} \mathcal{C} \xrightarrow{M \otimes_{\mathcal{C}} \Delta_{\mathcal{C}}} M \otimes_{\mathcal{C}} \mathcal{C} \otimes_A \mathcal{C} \xrightarrow{\bar{\mu}_M \otimes_A \mathcal{C}} M \otimes_A \mathcal{C}.$$

One can easily check that  $(M, \rho^M) \in \mathcal{M}^{\mathcal{C}}$ , thus we obtain a functor  $\Gamma : \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}^{\mathcal{C}}$  acting on the morphisms as the identity. We leave it to the reader to verify that  $\Xi \circ \Gamma$  and  $\Gamma \circ \Xi$  are the identity functors on  $\mathcal{M}_{\mathcal{C}}$  and  $\mathcal{M}^{\mathcal{C}}$  respectively. Symmetry arguments prove  ${}^{\mathcal{C}}\mathcal{M} \cong {}_{\mathcal{C}}\mathcal{M}$ .

To prove the final statement  $P \otimes_{\mathcal{C}} Q \cong P \otimes^{\mathcal{C}} Q$ , consider the following diagram, where the row and column represent respectively an equalizer and a coequalizer.

$$\begin{array}{ccccc}
 & & P \otimes_A \mathcal{C} \otimes_A Q & & \\
 & & \Downarrow & & \\
 P \otimes^{\mathcal{C}} Q & \xrightarrow{\iota} & P \otimes_A Q & \xRightarrow{\quad} & P \otimes_A \mathcal{C} \otimes_A Q \\
 & & \downarrow \pi & & \\
 & & P \otimes_{\mathcal{C}} Q & & 
 \end{array}$$

The map  $\alpha : P \otimes_A Q \rightarrow P \otimes_A Q$ ,  $\alpha(p \otimes_A q) := p^{[0]} \otimes_A p^{[1]} \cdot q$  satisfies, for all  $p \otimes_A q \in P \otimes_A Q$ ,

$$\begin{aligned}
 \alpha(p \otimes_A q) &= p^{[0]} \otimes_A p^{[1]} \cdot q = p^{[0]} \otimes_A \gamma(p^{[1]} \otimes_A q^{[-1]}) q^{[0]} \\
 &= p^{[0]} \gamma(p^{[1]} \otimes_A q^{[-1]}) \otimes_A q^{[0]} = p \cdot q^{[-1]} \otimes_A q^{[0]}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 ((\rho^P \otimes_A Q) \circ \alpha)(p \otimes_A q) &= p^{[0]} \otimes_A p^{[1]} \otimes_A p^{[2]} \cdot q \\
 &= p^{[0]} \otimes_A p^{[1]} \otimes_A \gamma(p^{[2]} \otimes_A q^{[-1]}) q^{[0]} \\
 &= p^{[0]} \otimes_A p^{[1]} \gamma(p^{[2]} \otimes_A q^{[-1]}) \otimes_A q^{[0]} \\
 &= p^{[0]} \otimes_A \gamma(p^{[1]} \otimes_A q^{[-2]}) q^{[-1]} \otimes_A q^{[0]} \\
 &= p^{[0]} \gamma(p^{[1]} \otimes_A q^{[-2]}) \otimes_A q^{[-1]} \otimes_A q^{[0]} \\
 &= p \cdot q^{[-2]} \otimes_A q^{[-1]} \otimes_A q^{[0]} = ((P \otimes_A \rho^Q) \circ \alpha)(p \otimes_A q),
 \end{aligned}$$

where we used (2.30) in the fourth equation. Therefore, we obtain by universality of the equalizer a unique morphism  $\alpha' : P \otimes_A Q \rightarrow P \otimes^{\mathcal{C}} Q$  such that  $\alpha = \iota \circ \alpha'$ . Furthermore,  $\alpha'$  is a left inverse for  $\iota$ , i.e. for all  $p \otimes_A q \in P \otimes^{\mathcal{C}} Q$ ,

$$(\alpha' \circ \iota)(p \otimes_A q) = p^{[0]} \otimes_A p^{[1]} \cdot q = p \otimes_A q^{[-1]} \cdot q^{[0]} = p \otimes_A q.$$

Next we check that  $\alpha' \circ (\mu_P \otimes_A Q) = \alpha' \circ (P \otimes_A \mu_Q)$ . Take any  $p \otimes_A c \otimes_A q \in P \otimes_A \mathcal{C} \otimes_A Q$ , then we find

$$\begin{aligned}
 \alpha(p \cdot c \otimes_A q) &= \alpha(p^{[0]} \gamma(p^{[1]} \otimes_A c) \otimes_A q) \\
 &= p^{[0]} \otimes_A \gamma(p^{[1]} \gamma(p^{[2]} \otimes_A c) \otimes_A q^{[-1]}) q^{[0]} \\
 &= p^{[0]} \otimes_A \gamma(\gamma(p^{[1]} \otimes_A c^{(1)}) c^{(2)} \otimes_A q^{[-1]}) q^{[0]} \\
 &= p^{[0]} \otimes_A \gamma(p^{[1]} \otimes_A c^{(1)}) \gamma(c^{(2)} \otimes_A q^{[-1]}) q^{[0]} \\
 &= p^{[0]} \otimes_A \gamma(p^{[1]} \otimes_A c^{(1)}) \gamma(c^{(2)} \otimes_A q^{[-1]}) q^{[0]} \\
 &= p^{[0]} \otimes_A \gamma(p^{[1]} \otimes_A \gamma(c \otimes_A q^{[-2]}) q^{[-1]}) q^{[0]} \\
 &= p^{[0]} \gamma(p^{[1]} \otimes_A \gamma(c \otimes_A q^{[-2]}) q^{[-1]}) \otimes_A q^{[0]} \\
 &= \alpha(p \otimes_A \gamma(c \otimes_A q^{[-1]}) q^{[0]}) \\
 &= \alpha(p \otimes_A c \cdot q).
 \end{aligned}$$

Since  $\alpha$  and  $\alpha'$  differ by the monomorphism  $\iota$ , from universal property of the coequalizer we therefore obtain a unique morphism  $\beta : P \otimes_{\mathcal{C}} Q \rightarrow P \otimes^{\mathcal{C}} Q$  such that  $\alpha' = \beta \circ \pi$ .



We then easily compute

$$\beta \circ \pi \circ \iota = \alpha' \circ \iota = P \otimes^{\mathcal{C}} Q \quad \text{and} \quad \pi \circ \iota \circ \beta \circ \pi = \pi \circ \iota \circ \alpha' = \pi \circ \alpha = \pi.$$

Since  $\pi$  is an epimorphism, latter identity implies  $\pi \circ \iota \circ \beta = P \otimes_{\mathcal{C}} Q$ . Hence  $\beta$  is an isomorphism with inverse  $\pi \circ \iota$ .  $\square$

As explained in the introduction, the following theorem improves [29, Corollary 9.4], [31, 5.7, 5.8], [11, Proposition 5.6] and is ultimately related to [27, Theorem I]. Let us emphasize that in the present version of the theorem no projectivity condition on the  $A$ -module  $\mathcal{C}$  is requested.

Note that if an  $R$ - $A$  bimodule  $\Sigma$  is an  $R$ -firmly projective right  $A$ -module then the  $R$ -bimodule map  $R \rightarrow \Sigma \otimes_A \Sigma^*$ ,  $r \mapsto \sum x_r \otimes_A \xi_r$  in Theorem 1.20 (iii) induces a (non-unital) ring map  $R \rightarrow S := \{ x\xi(-) \mid x \in \Sigma, \xi \in \Sigma^* \} \subseteq \text{End}_A(\Sigma)$ ,  $r \mapsto \sum x_r \xi_r(-)$ .

**Theorem 2.18.** *Let  $\mathcal{C}$  be a coseparable coring over a unital ring  $A$ ,  $R$  a firm ring and  $\Sigma \in {}_R\mathcal{M}^{\mathcal{C}}$ , such that  $\Sigma$  is an  $R$ -firmly projective right  $A$ -module. If  $R$  is a left ideal in  $T = \text{End}^{\mathcal{C}}(\Sigma)$ , then the following statements are equivalent.*

- (i)  $\text{can} : \Sigma^{\dagger} \otimes_R \Sigma \rightarrow \mathcal{C}$  is surjective;
- (ii)  $\text{can}$  is an isomorphism of  $A$ -coring;
- (iii)  $\text{Hom}(\Sigma, -) \otimes_R R : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_R$  is fully faithful;
- (iv)  $- \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}^{\mathcal{C}}$  is an equivalence of categories.

*Proof.* Obviously, (ii) implies (i) and (vi) implies (iii). The implication (iii)  $\Rightarrow$  (ii) follows from the structure Theorem for firm Galois comodules, see Corollary 2.3 (vi). We only have to prove that (i) implies (iv). Recall from Corollary 2.3 (v) that the functor  $- \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}^{\mathcal{C}}$  has a right adjoint  $- \otimes^{\mathcal{C}} \Sigma^{\dagger}$ . Applying Proposition 2.17, we obtain the following commutative diagram of functors.

$$(2.31) \quad \begin{array}{ccc} \mathcal{M}_R & \begin{array}{c} \xleftarrow{- \otimes^{\mathcal{C}} \Sigma^{\dagger}} \\ \xrightarrow{- \otimes_R \Sigma} \end{array} & \mathcal{M}^{\mathcal{C}} \\ & \begin{array}{c} \nwarrow - \otimes_R \Sigma \\ \searrow - \otimes^{\mathcal{C}} \Sigma^{\dagger} \end{array} & \begin{array}{c} \uparrow \Gamma \\ \downarrow \Xi \end{array} \\ & & \mathcal{M}_{\mathcal{C}} \end{array}$$

We know from Proposition 2.17 that the vertical functors describe an isomorphism of categories, hence the horizontal functors establish an equivalence if and only if the diagonal functors do so. The diagonal functors are obtained by tensor functors between two module categories and can thus be obtained from the Morita context  $\mathbb{C} = (R, \mathcal{C}, \Sigma, \Sigma^{\dagger}, \nabla, \blacktriangledown)$ , where the connecting maps are given by the formulae

$$\begin{aligned} \nabla : \Sigma \otimes_{\mathcal{C}} \Sigma^{\dagger} &\cong \Sigma \otimes^{\mathcal{C}} \Sigma^{\dagger} \cong \text{End}^{\mathcal{C}}(\Sigma) \otimes_R R \cong R; \\ \blacktriangledown = \text{can} : \Sigma^{\dagger} \otimes_R \Sigma &\rightarrow \mathcal{C}, \quad \xi \blacktriangledown x = \xi(x^{[0]})x^{[1]}. \end{aligned}$$

Since  $R$  is a left ideal in  $\text{End}^{\mathcal{C}}(\Sigma)$  by assumption, both connecting maps of  $\mathbb{C}$  are surjective. Hence the diagonal functors in (2.31) establish an equivalence by Theorem 1.12, what proves the claim.  $\square$

The following example provides another proof for [22, Corollary 4.2] and it also illustrates how our theory goes beyond the standard case.

**Example 2.19.** Let  $\iota : B \rightarrow A$  be a split extension of unital rings, i.e. such that there exists a  $B$ -linear morphism  $E : A \rightarrow B$  such that  $E \circ \iota = \text{id}_B$ . Then the Sweedler coring  $\mathcal{C} = A \otimes_B A$  is coseparable, with  $\mu : A \otimes_B A \otimes_A A \otimes_B A \cong A \otimes_B A \otimes_B A \rightarrow A \otimes_B A$  given by  $\mu(a \otimes_B a' \otimes_B a'') = aE(a') \otimes_B a''$  (see [8]). Furthermore, the category  $\mathcal{M}^{\mathcal{C}}$  is known to be isomorphic to the category  $\text{Desc}(A/B)$  of descent data associated to the ring extension  $\iota$ . Note that in this case  $\text{End}^{\mathcal{C}}(A) \cong \{ t \in A \mid t \otimes_B 1_A = 1_A \otimes_B t \}$  is equal to  $B$ , hence by Theorem 2.18 we find that the categories  $\mathcal{M}_B$  and  $\text{Desc}(A/B)$  are equivalent by the functor  $-\otimes_B A$ .

More generally, let  $\Sigma \in {}_B\mathcal{M}_A$  be a finitely generated projective right  $A$ -module which is separable in the sense that the evaluation map

$$(2.32) \quad \Sigma \otimes_A {}_B\text{Hom}(\Sigma, B) \rightarrow B, \quad x \otimes_A \xi \mapsto \xi(x)$$

is a split epimorphism of  $B$ -bimodules. Then the associated comatrix coring  $\mathcal{C} = \Sigma^* \otimes_B \Sigma$  is again coseparable (see [7]). Since in this case  $B \rightarrow \text{End}_A(\Sigma)$ ,  $b \mapsto (x \mapsto bx)$  is a split extension of unital rings, we conclude that  $B \cong \text{End}^{\mathcal{C}}(\Sigma)$ . Thus we find that the functor  $-\otimes_B \Sigma$  is an equivalence between  $\mathcal{M}_B$  and the category  $\text{Desc}(\Sigma)$  of generalized descent data.

Consider now a unital ring  $A$ , a firm ring  $R$  and an  $R$ -firmly projective right  $A$ -module  $\Sigma$ . Assume that  $\Sigma$  is a separable  $R$ - $A$  bimodule, i.e. replacing  $B$  by  $R$  in (2.32), we obtain a split epimorphism of  $R$ -bimodules. Then the corresponding comatrix  $A$ -coring  $\mathcal{C} := \Sigma^* \otimes_R \Sigma$  is coseparable. Indeed, similarly to [7, Theorem 3.5], a bilinear retraction of the coproduct in  $\mathcal{C}$  is given by

$$(\Sigma^* \otimes_R \Sigma) \otimes_A (\Sigma^* \otimes_R \Sigma) \rightarrow \Sigma^* \otimes_R \Sigma, \quad (\varphi \otimes_R z) \otimes_A (\psi \otimes_R y) \mapsto \sum \varphi \xi_r(z\psi(x_r)) \otimes_R {}^r y,$$

where  $r \mapsto \sum x_r \otimes_A \xi_r$  is an  $R$ -bimodule retraction of (2.32).

Note that  $R$  is a left ideal in  $\text{End}^{\mathcal{C}}(\Sigma)$ . Indeed, taking into account the explicit form  $y \mapsto \sum e_r \otimes_A f_r \otimes_R {}^r y$  of the  $\mathcal{C}$ -coaction on  $\Sigma$ , given in terms of the map  $R \rightarrow \Sigma \otimes_A \Sigma^*$ ,  $r \mapsto \sum e_r \otimes_A f_r$ , encoding  $R$ -firm projectivity of the right  $A$ -module  $\Sigma$ , it follows that  $\Phi \in \text{End}_A(\Sigma)$  is a right  $\mathcal{C}$  comodule map if and only if

$$(2.33) \quad \sum e_r \otimes_A f_r \otimes_R {}^r \Phi(y) = \sum \Phi(e_r) \otimes_A f_r \otimes_R {}^r y,$$

for all  $y \in \Sigma$ . Applying the map

$$\Sigma \otimes_A \Sigma^* \otimes_R \Sigma \rightarrow \Sigma \quad z \otimes_A \psi \otimes_R y \mapsto \sum \xi_r(z\psi(x_r)) {}^r y$$

to both sides of (2.33), we conclude that

$$\Phi(y) = \sum (\xi_{r'} \circ \Phi)(rx_{r'}) {}^{r'} y = \sum (\xi_r \circ \Phi)(x_r) {}^r y,$$

where the last equality follows by the  $R$ -bilinearity condition  $\sum rx_{r'} \otimes_A \xi_{r'} = x_r \otimes_A \xi_r {}^{r'}$ . This shows that  $\Phi r = \xi_r(\Phi(x_r)) \in R$ , hence  $R$  is a left ideal in  $\text{End}^{\mathcal{C}}(\Sigma)$ , as stated.

The canonical map corresponding to the  $R$ - $\mathcal{C}$  bicomodule  $\Sigma$  is the identity map, so we may conclude by Theorem 2.18 that  $-\otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}^{\mathcal{C}}$  is an equivalence functor.

**Example 2.20.** Let  $H$  be a Hopf algebra over a commutative ring  $k$ . Then  $H \otimes_k H$  admits the structure of a coseparable  $H$ -coring cf. [4, 8.8]. Right comodules of the  $H$ -coring  $H \otimes_k H$  are known as *Hopf modules* of  $H$ . Their category is denoted by  $\mathcal{M}_H^H$ .

Let  $\Sigma$  be an  $H$ -Hopf module and  $T$  be the algebra of Hopf module endomorphisms of  $\Sigma$ . Assume that there is a firm ring  $R$  which is a left ideal in  $T$ . If  $\Sigma$  is an  $R$ -firmly projective right  $A$ -module then, by Theorem 2.18, the functor  $- \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}_H^H$  is an equivalence if and only if the canonical map  $\text{can} : \Sigma^* \otimes_R \Sigma \rightarrow H \otimes_k H$  is surjective.

Choose in particular  $\Sigma = H$  (with  $H$ -action given by the multiplication, and  $H$ -coaction given by comultiplication) and  $R = T \cong k$ . Then the inverse of  $\text{can}$  is easily constructed in terms of the antipode of  $H$  (see e.g. [9, 15.5]). Thus we obtain an alternative proof of the claim in (iv), what is usually referred to as the *Fundamental Theorem of Hopf modules*.

#### ACKNOWLEDGEMENT

The first author acknowledges a Bolyai János Research Scholarship and financial support of the Hungarian Scientific Research Fund OTKA F67910.

The second author thanks the Fund for Scientific Research–Flanders (Belgium) (F.W.O.–Vlaanderen) for a Postdoctoral Fellowship.

#### REFERENCES

- [1] J. Abuhlail, *Morita contexts for corings and equivalences*, in: ‘Hopf algebras in noncommutative geometry and physics’. S. Caenepeel and F. Van Oystaeyen (eds.), Marcel Dekker 2005, pp. 1–29.
- [2] H. Bass, *Algebraic K-theory*. Benjamin, New York, 1968.
- [3] M. Beattie, S. Dăscălescu and Ş. Raianu, *Galois extensions for co-Frobenius Hopf algebras*, J. Algebra **198** (1997), 164–183.
- [4] G. Böhm, T. Brzeziński and R. Wisbauer, *Monads and comonads in module categories*, Preprint arXiv:0804.1460.
- [5] G. Böhm and J. Vercruysse, *Morita theory for coring extensions and cleft bicomodules*, Adv. Math. **209** (2007), 611–648. *Corrigendum*, to be published. See also arXiv:math/0601464v2.
- [6] T. Brzeziński, *A note on coring extensions*, Ann. Univ. Ferrara - Sez. VII - Sc. Mat. 51 (2005), 15–27. A corrected version is available at arXiv:math/0410020v3.
- [7] T. Brzeziński and J. Gómez Torrecillas, *On comatrix corings and bimodules*, K-Theory, **29** (2003), 101–115.
- [8] T. Brzeziński, L. Kadison and R. Wisbauer, *On coseparable and biseparable corings*, in: ‘Hopf algebras in noncommutative geometry and physics’. S. Caenepeel and F. Van Oystaeyen (eds.), Marcel Dekker 2005, pp. 71–87.
- [9] T. Brzeziński and R. Wisbauer, *Corings and Comodules*. Cambridge University Press, Cambridge, 2003.
- [10] S. Caenepeel, *Brauer Groups, Hopf algebras and Galois theory*. K-monographs in Mathematics, Kluwer Academic Publishers, Dordrecht, 1998.
- [11] S. Caenepeel, E. De Groot and J. Vercruysse, *Galois theory for comatrix corings: descent theory, Morita theory, Frobenius and separability properties*, Trans. Amer. Math. Soc. **359** (2007), 185–226.
- [12] S. Caenepeel, M. Iovanov, *Comodules over semiperfect corings*, in “Proceedings of the International Conference on Mathematics and its Applications, ICMA 2004”, S.L. Kalla and M.M. Chawla (Eds.), Kuwait University, Kuwait, 2005, 135–160.
- [13] S. Caenepeel, J. Vercruysse and S. Wang, *Rationality properties for Morita contexts associated to corings*, in “Hopf algebras in non-commutative geometry and physics”, Caenepeel S. and Van Oystaeyen, F. (eds.), Lect. Notes Pure Appl. Math., Dekker, New York (2005) 113–136.
- [14] F. Castaño-Iglesias and J. Gómez-Torrecillas, *Wide Morita contexts*, Comm. Algebra **23** (2) (1995), 601–622.
- [15] N. Chifan, S. Dăscălescu and C. Năstăsescu, *Wide Morita contexts, relative injectivity and equivalence results*, J. Algebra **284** (2005), 705–736.
- [16] M. Cohen, D. Fischman and S. Montgomery, *Hopf Galois extensions, smash products, and Morita equivalence*, J. Algebra **133** (1990), 351–372.

- [17] Y. Doi, *Generalised smash products and Morita contexts for arbitrary Hopf algebras*, in: ‘Advances in Hopf Algebras’ J. Bergen and S. Montgomery (eds.), Marcel Dekker 1994.
- [18] L. El Kaoutit and J. Gómez-Torrecillas, *Morita Duality for Corings over Quasi-Frobenius Rings*, in: ‘Hopf algebras in noncommutative geometry and physics’. S. Caenepeel and F. Van Oystaeyen (eds.), Marcel Dekker 2005, pp. 137–153.
- [19] J. Gómez-Torrecillas and J. Vercruysse, *Comatrix corings and Galois comodules over firm rings*, *Algebr. Represent. Theory*, **10** (3) (2007), 271–306.
- [20] F. Grandjean and E. M. Vitale, *Morita equivalence for regular algebras*, *Cahiers Topologie Géom. Différentielle Catég.* **39** (1998), 137–153.
- [21] M. Iovanov and J. Vercruysse, *Co-Frobenius Corings and Related Functors*, *J. Pure Appl. Algebra* (2008), doi:10.1016/j.jpaa.2007.11.015, in press.
- [22] G. Janelidze and W. Tholen, *Facets of descent III. Monadic descent for rings and algebras*, *Appl. Categ. Structures* **12** (2004), 461–477.
- [23] T. Kato and K. Ohtake, *Morita contexts and equivalences*, *J. Algebra* **61** (1979), 360–366.
- [24] L. Marín, *Morita equivalence based on contexts for various categories of modules over associative rings*, *J. Pure Appl. Algebra* **133** (1998), 219–232.
- [25] B. J. Müller, *The quotient category of a Morita context*, *J. Algebra* **28** (1974), 389–407.
- [26] D. Quillen, *Module theory over nonunital rings*, unpublished notes, 1997.
- [27] H.-J. Schneider, *Principal homogeneous spaces for arbitrary Hopf algebras*, *Israel J. Math.* **72** (1990), 167–195.
- [28] J. Vercruysse, *Equivalences between categories of modules and categories of comodules*, *Acta Math. Sin. (Engl. Ser.)* (2008), in press.
- [29] J. Vercruysse, *Galois Theory for Corings and Comodules*, PhD thesis. Vrije Universiteit Brussel, 2007.
- [30] R. Wisbauer, *On the category of comodules over corings*, in: *Mathematics & Mathematics Education* (Bethlehem, 2000), pp 325–336, World Sci. Publ., River Edge, NJ, 2002.
- [31] R. Wisbauer, *On Galois comodules*, *Comm. Algebra* **34** (2006), 2683–2711.

RESEARCH INSTITUTE FOR PARTICLE AND NUCLEAR PHYSICS, BUDAPEST,  
H-1525 BUDAPEST 114, P.O.B. 49, HUNGARY  
*E-mail address:* G.Bohm@rmki.kfki.hu

VRIJE UNIVERSITEIT BRUSSEL VUB, PLEINLAAN 2, B-1050, BRUSSEL, BELGIUM  
*E-mail address:* joost.vercruysse@vub.ac.be